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STRONG BOUNDED SOLUTIONS FOR NONLINEAR PARABOLIC SYSTEMS

NSOKI MAVINGA, MUBENGA N. NKASHAMA

Abstract. In this article we study the existence of strong bounded solutions for nonlinear parabolic systems on a domain which is bounded in space and unbounded in time (namely the entire real line). We use nonlinear iteration arguments combined with some a priori estimates to derive the existence results. We also provide conditions under which we have a positive solution. Some examples are given to illustrate the results.

1. Introduction

We consider the nonlinear parabolic system

$$
\begin{gathered}
\frac{\partial u}{\partial t}(x,t) - L_1 u(x,t) = f_1(x,t,u,v) \quad \text{in } \Omega \times \mathbb{R}, \\
\frac{\partial v}{\partial t}(x,t) - L_2 v(x,t) = f_2(x,t,u,v) \quad \text{in } \Omega \times \mathbb{R}, \\
\mathcal{B}_1 u(x,t) = g_1(x,t,u,v) \quad \text{on } \partial\Omega \times \mathbb{R}, \\
\mathcal{B}_2 v(x,t) = g_2(x,t,u,v) \quad \text{on } \partial\Omega \times \mathbb{R}, \\
\sup_{\Omega\times\mathbb{R}} \{|u(x,t)|, |v(x,t)|\} < \infty,
\end{gathered}
\tag{1.1}
$$

where Ω is a bounded, open and connected subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and closure $\overline{\Omega}$. We suppose that L_k are second order, uniformly elliptic differential operators with time-dependent coefficients and $\mathcal{B}_k$ are linear first-order boundary operators which are either Dirichlet, Neumann or regular oblique type. We suppose that the coefficients of the operators L_k and $\mathcal{B}_k$ are measurable and bounded. The reaction and the boundary nonlinearities f_k and g_k are, say, Carathéodory functions.

We are interested in bounded solutions existing for all time. Steady-state, time-periodic solutions and (bounded) attractors as well as almost-periodic solutions are only a few examples of solutions existing for all times, see e.g. [5, 6, 9, 18, 23]. The study of nonlinear parabolic systems for large time have important applications in ecology. Full bounded solutions are thus important in both backward and forward

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dynamics. Many papers have been devoted to the study of nonlinear parabolic systems with given initial conditions. For some recent results in this direction and bibliography we refer to [2, 3, 14, 19, 4, 9, 18, 23], and others. However, to the best of our knowledge, not much seems to be done for system (1.1) for the case of nonlinear boundary conditions and in the unbounded time-domain (namely, the entire real line). A few results were obtained in the scalar case by the authors [15, 16]. Since we are dealing with nonlinear parabolic systems with nonlinear boundary conditions and without initial conditions, many of the tools used for compact or semi-infinite time interval such as the maximum principle and the fixed-point results are not directly applicable. Thus, the need to develop new tools for studying the problem. Moreover, we deduce the comparison principle which is valid on the entire real line in time and used some nonlinear iteration arguments to obtain the existence results.

The paper is organized as follows. In Section 2, we formulate general assumptions which are needed throughout, and state our main results concerning the existence of bounded solutions existing for all times for nonlinear systems with (possibly) nonlinear boundary conditions. We assume that the nonlinearities in the reaction and on the boundary satisfy some growth conditions. In Section 3, we state some results on (scalar) linear parabolic equations which are needed in the proof of our main results. In Section 4, we prove the main results. We conclude the paper with some examples which illustrate our results.

2. Assumptions and Main results

All functions in this paper will take values in $\mathbb{R}$ and all vector spaces are over the reals. We assume that Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. We assume that $\partial\Omega$ belongs to C^2; $\mu \in (0,1)$ and $p = (N+2)/(1-\mu)$. We consider the second order parabolic operators in $\Omega \times \mathbb{R}$ given by

$$\frac{\partial u}{\partial t} - L_k u, \tag{2.1}$$

where

$$L_k u := \sum_{i,j=1}^{N} a_{ij}^{(k)}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^{(k)}(x,t) \frac{\partial u}{\partial x_i} + c^{(k)}(x,t) u, \quad k = 1, 2$$

with symmetric positive definite coefficient-matrices $(a_{ij}^{(k)})$. We assume that

- (i) $a_{ij}^{(k)} \in C(\overline{\Omega} \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$, $b_i^{(k)}, c^{(k)} \in L^\infty(\Omega \times \mathbb{R})$.
- (ii) There are constants $c_0 \geq 0$ and $\gamma_0 > 0$ such that for a.e. $(x,t) \in \overline{\Omega} \times \mathbb{R}$, $c^{(k)}(x,t) \leq -c_0$ and $\sum_{i,j=1}^N a_{ij}^{(k)}(x,t)\xi_i\xi_j \geq \gamma_0 |\xi|^2$ for all $\xi \in \mathbb{R}^N$.

Let ϵ denote a variable which assumes the values 0 and 1 only. We define the boundary operators $\mathcal{B}_{k,\epsilon}$ by

$$\mathcal{B}_{k,\epsilon} u := \epsilon \frac{\partial u}{\partial \nu} + \alpha^{(k)}(x,t) u, \quad k = 1, 2 \tag{2.2}$$

where $\alpha^{(k)} \in W^{1,1/2}_{p,\mathrm{loc}}(\partial\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$, and for all $(x,t) \in \partial\Omega \times \mathbb{R}$, $\alpha^{(k)}(x,t) \geq \alpha_0 \geq 0$. The constant α_0 is such that $\alpha_0 > 0$ if $\epsilon = 0$, and $\alpha_0 \geq 0$ if $\epsilon = 1$. Moreover,

$$c_0 + \alpha_0 > 0, \tag{2.3}$$

which implies that the coefficients $c^{(k)}(x,t)$ and $\alpha^{(k)}(x,t)$ do not vanish simultaneously. Thus, for $\epsilon = 0$, $\mathcal{B}_{k,0}u$ is a Dirichlet boundary condition whereas for $\epsilon = 1$, $\mathcal{B}_{k,1}u$ corresponds to a Neumann or a regular oblique derivative boundary condition.

In what follows, the inequality $(u_1, v_1) \le (u_2, v_2)$ means that $u_1 \le u_2$ and $v_1 \le v_2$. The functions f_k, g_k depend in general on u and v; except for the Dirichlet boundary condition where g_k are independent of u and v; that is, $g_k(x,t,u,v) = g_k(x,t)$. The reaction functions $f_k \in L^\infty_{\rm car}(\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ are L^∞-Carathéodory functions; that is,

(A1) $f_k(\cdot,\cdot,u,v)$ is measurable; $f_k(x,t,\cdot,\cdot)$ is continuous for a.e. $(x,t) \in \Omega \times \mathbb{R}$; and for every $r > 0$ there is a function $M_{k,r} \in L^\infty(\Omega \times \mathbb{R})$ such that $|f_k(x,t,u,v)| \le M_{k,r}(x,t)$ for a.e. $(x,t) \in \Omega \times \mathbb{R}$ and all $(u,v) \in [-r,r] \times [-r,r]$.

The boundary functions g_k are also L^∞-Carathéodory functions; i.e, they satisfy (A1), but in addition if $\epsilon = 0$ (i.e Dirichlet boundary condition) then $g_k \in W^{2-1/p,(2-1/p)/2}_{p,\rm loc}(\partial\Omega \times \mathbb{R})$ and if $\epsilon = 1$ (i.e Neumann boundary condition) then $g_k(x,t,u,v)$ satisfy the Lipschitz condition in $J_1 \times J_2 \subset \mathbb{R}^2$, uniformly in $(x,t) \in \overline{\Omega} \times \mathbb{R}$, where J_1 and J_2 are closed intervals in $\mathbb{R}$; that is

(A2) for every $J_1 \times J_2 \subset \mathbb{R} \times \mathbb{R}$, there is a constant $\varrho_k = \varrho_k(\partial\Omega \times \mathbb{R} \times J_1 \times J_2) > 0$ such that
$$\begin{aligned}&|g_k(x,t,u_1,v_1) - g_k(y,s,u_2,v_2)|\\ &\le \varrho_k \left[|x-y|^2 + |t-s| + |u_1-u_2|^2 + |v_1-v_2|^2\right]^{1/2}\end{aligned}$$
for all $(x,t,u_1,v_1), (y,s,u_2,v_2) \in \partial\Omega \times \mathbb{R} \times J_1 \times J_2$.

(A3) The vector functions $\mathbf{f} = (f_1, f_2), \mathbf{g} = (g_1, g_2)$ are quasimonotone in $J_1 \times J_2$, that is, they satisfy one of the following quasimonotonicity properties: (f_1, f_2), (g_1, g_2) are quasimonotone nondecreasing (quasimonotone nonincreasing) in $J_1 \times J_2$; i.e.,
$$\text{for fixed } u \in J_1, f_1, g_1 \text{ are nondecreasing (nonincreasing) in } v \in J_2,$$
and
$$\text{for fixed } v \in J_2, f_2, g_2 \text{ are nondecreasing (nonincreasing) in } u \in J_1.$$

We wish to emphasize the fact that this 'additional' local Lipschitz condition (A2) on the boundary nonlinearities $g_k(x,t,u,v)$ is needed to obtain *a priori* estimates for the *boundary traces* of solutions. It ensures that the boundary superposition (Nemytskĭi) operator associated with the function $g_k(x,t,\cdot,\cdot)$ maps $W^{1-1/p,(1-1/p)/2}_{p,\rm loc}(\partial\Omega\times\mathbb{R})\times W^{1-1/p,(1-1/p)/2}_{p,\rm loc}(\partial\Omega\times\mathbb{R})$ into $W^{1-1/p,(1-1/p)/2}_{p,\rm loc}(\partial\Omega\times\mathbb{R})$; the latter is the condition needed on the boundary data to get strong solutions, i.e., solutions in $W^{2,1}_{p,\rm loc}(\Omega\times\mathbb{R}) \times W^{2,1}_{p,\rm loc}(\Omega\times\mathbb{R})$. In particular, for $g_k(x,t,u,v) = g_k(x,t)$ independent of (u,v), it implies that $g_k \in W^{1-1/p,(1-1/p)/2}_{p,\rm loc}(\partial\Omega \times \mathbb{R})$.

Based on the type of quasimonotonicity property, we will use the following definitions for strong sub and supersolutions.

Definition 2.1. A pair of functions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ in $W^{2,1}_{p,\rm loc}(\Omega\times\mathbb{R})\times W^{2,1}_{p,\rm loc}(\Omega\times\mathbb{R})$ are ordered *subsolution* and *supersolution* of the system (1.1) if

(1) $(\underline{u}, \underline{v}) \le (\overline{u}, \overline{v})$, that is, $\underline{u} \le \overline{u}$ and $\underline{v} \le \overline{v}$, and one of the following conditions holds.

(2) When (f_1, f_2) and (g_1, g_2) are quasimonotone nondecreasing

$$\begin{gathered}
\frac{\partial \overline{u}}{\partial t} - L_1\overline{u} - f_1(x,t,\overline{u},\overline{v}) \geq 0 \geq \frac{\partial \underline{u}}{\partial t} - L_1\underline{u} - f_1(x,t,\underline{u},\underline{v}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial \overline{v}}{\partial t} - L_2\overline{v} - f_2(x,t,\overline{u},\overline{v}) \geq 0 \geq \frac{\partial \underline{v}}{\partial t} - L_2\underline{v} - f_2(x,t,\underline{u},\underline{v}) \quad \text{in } \Omega\times\mathbb{R},\\
\mathcal{B}_1\overline{u} - g_1(x,t,\overline{u},\overline{v}) \geq 0 \geq \mathcal{B}_1\underline{u} - g_1(x,t,\underline{u},\underline{v}) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\mathcal{B}_2\overline{v} - g_2(x,t,\overline{u},\overline{v}) \geq 0 \geq \mathcal{B}_2\underline{v} - g_2(x,t,\underline{u},\underline{v}) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}(|\underline{u}|,|\underline{v}|,|\overline{u}|,|\overline{v}|) < \infty.
\end{gathered}$$

(3) When (f_1, f_2) and (g_1, g_2) are quasimonotone nonincreasing

$$\begin{gathered}
\frac{\partial \overline{u}}{\partial t} - L_1\overline{u} - f_1(x,t,\overline{u},\underline{v}) \geq 0 \geq \frac{\partial \underline{u}}{\partial t} - L_1\underline{u} - f_1(x,t,\underline{u},\overline{v}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial \overline{v}}{\partial t} - L_2\overline{v} - f_2(x,t,\underline{u},\overline{v}) \geq 0 \geq \frac{\partial \underline{v}}{\partial t} - L_2\underline{v} - f_2(x,t,\overline{u},\underline{v}) \quad \text{in } \Omega\times\mathbb{R},\\
\mathcal{B}_1\overline{u} - g_1(x,t,\overline{u},\underline{v}) \geq 0 \geq \mathcal{B}_1\underline{u} - g_1(x,t,\underline{u},\overline{v}) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\mathcal{B}_2\overline{v} - g_2(x,t,\underline{u},\overline{v}) \geq 0 \geq \mathcal{B}_2\underline{v} - g_2(x,t,\overline{u},\underline{v}) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}(|\underline{u}|,|\underline{v}|,|\overline{u}|,|\overline{v}|) < \infty.
\end{gathered}$$

For the rest of this article, we assume that the interval $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}] \subseteq J_1\times J_2$.

Our main result for the system (1.1) is given by the following theorem.

Theorem 2.2. *Assume* (A1)—(A3) *are satisfied and suppose that* (1.1) *has an ordered subsolution* $(\underline{u},\underline{v})$ *and supersolution* $(\overline{u},\overline{v})$ *and* (f_1, f_2) *and* (g_1, g_2) *are quasimonotone nondecreasing (quasimonotone nonincreasing) in* $[(\underline{u},\underline{v}),(\overline{u},\overline{v})]$*. Then the system* (1.1) *has at least one solution* $(u,v) \in W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R})\times W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R})$ *such that*

$$(\underline{u},\underline{v}) \leq (u,v) \leq (\overline{u},\overline{v}) \text{ in } \overline{\Omega}\times\mathbb{R}.$$

As an immediate consequence of Theorem 2.2, we have the following corollary on the existence of positive full bounded solutions.

Corollary 2.3 (Positive Solutions)**.** *Assume that the assumptions in Theorem 2.2 are satisfied. Suppose that either* (f_1, f_2) *and* (g_1, g_2) *are quasimonotone nondecreasing and*

$$f_1(x,t,0,0) \geq 0, \quad f_2(x,t,0,0) \geq 0, \quad g_1(x,t,0,0) \geq 0, \quad g_2(x,t,0,0) \geq 0.$$

When either (f_1, f_2) *and* (g_1, g_2) *are quasimonotone nonincreasing and*

$$f_1(x,t,0,v) \geq 0,\ f_2(x,t,u,0) \geq 0,\ g_1(x,t,0,v) \geq 0,\ \text{and } g_2(x,t,u,0) \geq 0.$$

Furthermore, assume that there exists a nonegative supersolution $(\overline{u},\overline{v})$ *of* (1.1)*. Then the system* (1.1) *has a nonnegative solution* $(u,v) \in W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R})\times W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R})$ *such that* $(u,v) \leq (\overline{u},\overline{v})$.

Indeed, observe that $(0,0)$ is a subsolution of the system (1.1). Therefore by Theorem 2.2, the system (1.1) has a nonnegative solution.

3. Auxiliary results and proof of the main result

To prove the main result stated in the previous section, we need some auxiliary results on scalar linear parabolic equations in $\Omega \times \mathbb{R}$. We refer to [15, 16] for the proof of these results.

Consider the linear boundary value problem

$$\begin{gathered} \frac{\partial u}{\partial t} - Lu = f \quad \text{in } \Omega \times \mathbb{R}, \\ \mathcal{B}_\epsilon u = \varphi \quad \text{on } \partial\Omega \times \mathbb{R}, \\ \sup_{\Omega \times \mathbb{R}} |u(x,t)| < \infty, \end{gathered} \tag{3.1}$$

where L is a second order, time dependent, uniformly elliptic differential operators and $\mathcal{B}_\epsilon$ is a first order boundary operator as defined in the previous section.

Proposition 3.1 (A priori estimates)**.** *Let $u \in W^{2,1}_{p,\mathrm{loc}}(\Omega \times \mathbb{R})$ be (uniformly) bounded at $-\infty$. Then there exists a constant K such that*

$$\sup_{\Omega\times\mathbb{R}} |u_\pm| \le K\Big(\sup_{\Omega\times\mathbb{R}} \big|\big(\frac{\partial u}{\partial t} - Lu\big)_\pm\big| + \sup_{\partial\Omega\times\mathbb{R}} |(\mathcal{B}_\epsilon u)_\pm| \Big); \tag{3.2}$$

which implies that

$$\sup_{\Omega\times\mathbb{R}} |u| \le K\Big(\sup_{\Omega\times\mathbb{R}} |\frac{\partial u}{\partial t} - Lu| + \sup_{\partial\Omega\times\mathbb{R}} |\mathcal{B}_\epsilon u| \Big).$$

The constant K depends only on the dimension N, the parabolicity constant γ_0, $\mathrm{diam}(\Omega)$*, and the L^∞-bounds of the coefficients of the operators L and $\mathcal{B}_\epsilon$.*

We deduce from the above proposition the following (weak) maximum type-comparison principle.

Corollary 3.2. (Weak Maximum/Comparison Principle) *Suppose that the conditions of Proposition 3.1 are met. Assume that $\frac{\partial u}{\partial t} - Lu \ge 0$ a.e. in $\Omega \times \mathbb{R}$ and that $\mathcal{B}_\epsilon u \ge 0$ on $\partial\Omega \times \mathbb{R}$. Then $u \ge 0$ in $\overline{\Omega} \times \mathbb{R}$.*

The next proposition deals with the existence result for linear parabolic equations.

Proposition 3.3. *Suppose that $f \in L^\infty(\Omega \times \mathbb{R})$ and $\varphi \in \mathrm{W}^{2-\epsilon-\frac{1}{p},(2-\epsilon-\frac{1}{p})/2}_{p,\mathrm{loc}}(\partial\Omega \times \mathbb{R}) \cap L^\infty(\partial\Omega \times \mathbb{R})$ with $p = \frac{N+2}{1-\mu}$. Then the problem (3.1) has a unique solution $u \in W^{2,1}_{p,\mathrm{loc}}(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$.*

Lemma 3.4 (Interpolation inequalities)**.** *Let $\Omega \times I \subset \mathbb{R}^n \times \mathbb{R}$ and $1 \le p < \infty$. There is a constant $C > 0$ such that for all $u \in W^{2,1}_p(\Omega \times I)$ one has*

$$|u|_{W^{1,1/2}_p(\Omega\times I)} \le C |u|^{1/2}_{W^{2,1}_p(\Omega\times I)} |u|^{1/2}_{L^p(\Omega\times I)}.$$

Moreover, for every $\varepsilon > 0$,

$$|u|_{W^{1,1/2}_p(\Omega\times I)} \le C\Big(\varepsilon |u|_{W^{2,1}_p(\Omega\times I)} + \frac{1}{4\varepsilon} |u|_{L^p(\Omega\times I)}\Big). \tag{3.3}$$

Proposition 3.5. *Consider the nonlinear parabolic boundary value problem*

$$\begin{gathered}\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = f(x,t,u) \quad \text{a.e. in } \Omega\times\mathbb{R},\\ \mathcal{B}u = \varphi(x,t,u) \quad \text{a.e. on } \partial\Omega\times\mathbb{R},\\ \sup_{\Omega\times\mathbb{R}} |u(x,t)| < \infty.\end{gathered} \tag{3.4}$$

Suppose that $\underline{u}, \overline{u}$ are ordered sub-solution and super-solution of (3.4) *and suppose that f and φ are nonincreasing in u, for $u \in [\underline{u}, \overline{u}]$. Then* (3.4) *has at most one solution u such that $\underline{u} \le u \le \overline{u}$.*

Lemma 3.6. *Let f satisfy* (A1)*; that is, $f \in L^\infty_{\rm car}(\Omega\times\mathbb{R}\times\mathbb{R})$. Then, for every $r \in \mathbb{R}$ with $r > 0$, there is a continuous function $m : [-r,r]\times[-r,r] \to \mathbb{R}$ such that $m(\cdot, v)$ is nondecreasing on $[-r,r]$, $m(u,\cdot)$ is nonincreasing on $[-r,r]$, $m(u,v) = -m(v,u)$ on $[-r,r]\times[-r,r]$, and*

$$\sup_{\Omega\times\mathbb{R}} |f(x,t,u) - f(x,t,v)| \le m(u,v) \tag{3.5}$$

for all $u, v \in [-r,r]$ with $u \ge v$.

Proposition 3.7. *Let* (A1)–(A3) *and the following condition hold,*

(LL) *The functions f_k $(k = 1,2)$ satisfy the one-sided Lipschitz condition in $J_1 \times J_2 \subset \mathbb{R}^2$, uniformly a.e. in $(x,t) \in \overline{\Omega}\times\mathbb{R}$; that is, there are constants $\theta_k \ge 0$ such that for every $(u_1,v), (u_2,v), (u,v_1), (u,v_2)$ in $J_1\times J_2$,*

$$\begin{gathered}f_1(x,t,u_1,v) - f_1(x,t,u_2,v) \ge -\theta_1(u_1-u_2), \quad \text{for } u_1 \ge u_2,\\ f_2(x,t,u,v_1) - f_2(x,t,u,v_2) \ge -\theta_2(v_1-v_2), \text{for } v_1 \ge v_2.\end{gathered}$$

Suppose that (1.1) *has an ordered subsolution $(\underline{u},\underline{v})$ and supersolution $(\overline{u},\overline{v})$ and (f_1,f_2) and (g_1,g_2) are quasimonotone nondecreasing (quasimonotone nonincreasing) in $[(\underline{u},\underline{v}),(\overline{u},\overline{v})]$. Then the system* (1.1) *has at least one solution $(u,v) \in W^{2,1}_{p,\rm loc}(\overline{\Omega}\times\mathbb{R}) \times W^{2,1}_{p,\rm loc}(\overline{\Omega}\times\mathbb{R})$ such that*

$$(\underline{u},\underline{v}) \le (u,v) \le (\overline{u},\overline{v}) \text{ in } \overline{\Omega}\times\mathbb{R}.$$

Proof. Let $\delta = \max\{\theta_1,\theta_2,\rho_1,\rho_2\}$. Consider the following modified problem.

$$\begin{gathered}\frac{\partial u}{\partial t} - L_1 u + \delta u = f_1(x,t,u,v) + \delta u \quad \text{in } \Omega\times\mathbb{R},\\ \frac{\partial v}{\partial t} - L_2 v + \delta v = f_2(x,t,u,v) + \delta v \quad \text{in } \Omega\times\mathbb{R},\\ \mathcal{B}_1 u + \delta u = g_1(x,t,u,v) + \delta u \quad \text{on } \partial\Omega\times\mathbb{R},\\ \mathcal{B}_2 v + \delta v = g_2(x,t,u,v) + \delta v \quad \text{on } \partial\Omega\times\mathbb{R},\\ \sup_{\Omega\times\mathbb{R}}\{|u(x,t)|, |v(x,t)|\} < \infty.\end{gathered} \tag{3.6}$$

To prove the existence of solutions for problem (1.1), it suffices to show that the modified problem (3.6) has a solution.

First, we construct a sequence (u_n, v_n) from the (linear) iteration process

$$
\begin{aligned}
\frac{\partial u_n}{\partial t} - L_1 u_n + \delta u_n &= f_1(x,t,u_{n-1},v_{n-1}) + \delta u_{n-1} \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial v_n}{\partial t} - L_2 v_n + \delta v_n &= f_2(x,t,u_{n-1},v_{n-1}) + \delta v_{n-1} \quad \text{in } \Omega\times\mathbb{R},\\
\mathcal{B}_1 u_n + \delta u_n &= g_1(x,t,u_{n-1},v_{n-1}) + \delta u_{n-1} \quad \text{on } \partial\Omega\times\mathbb{R},\\
\mathcal{B}_2 v_n + \delta v_n &= g_2(x,t,u_{n-1},v_{n-1}) + \delta v_{n-1} \quad \text{on } \partial\Omega\times\mathbb{R},\\
&\sup_{\Omega\times\mathbb{R}}\{|u_n(x,t)|, |v_n(x,t)|\} < \infty,
\end{aligned}
\tag{3.7}
$$

where the initial iteration (u_0, v_0) is determined by the quasimonotonicity property considered.

(i) If f_k and g_k are quasimonotone nondecreasing then we take either $(u_0,v_0) = (\underline{u},\underline{v})$ or $(u_0,v_0) = (\overline{u},\overline{v})$, and we denote the sequences constructed from the two initial iterations by $(\underline{u}_n,\underline{v}_n)$ and $(\overline{u}_n,\overline{v}_n)$, respectively.

(ii) If f_k and g_k are quasimonotone nonincreasing then we choose either $(u_0,v_0) = (\underline{u},\overline{v})$ or $(u_0,v_0) = (\overline{u},\underline{v})$, and we denote the sequences constructed from the two initial iterations by $(\underline{u}_n,\overline{v}_n)$ and $(\overline{u}_n,\underline{v}_n)$, respectively. For sake of discussion, we will present the rest of the proof for the case of quasimonotone nondecreasing functions. Similar arguments can be used for the quasimonotone nonincreasing case.

Observe that for each $n\in\mathbb{N}$, the above system consists of two linear uncoupled problems and $f_k(.,.,u_0,v_0)\in L^\infty(\Omega\times\mathbb{R})$ and $g_k(.,.,u_0,v_0)\in \mathrm{W}^{2-\epsilon-\frac{1}{p},(2-\epsilon-\frac{1}{p})/2}_{p,\mathrm{loc}}(\partial\Omega\times\mathbb{R})\cap L^\infty(\partial\Omega\times\mathbb{R})$ whenever $(u_0,v_0)\in[\underline{u},\overline{u}]\times[\underline{u},\overline{u}]$. Set $(u_0,v_0)=(\underline{u},\underline{v})$ then it follows from proposition 3.3 that problem (3.7) has a unique solution $(\underline{u}_1,\underline{v}_1)\in \mathrm{W}^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})\times \mathrm{W}^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})$. Moreover if $(u_0,v_0) = (\overline{u},\overline{v})$ then a similar argument shows that problem (3.7) has a unique solution $(\overline{u}_1,\overline{v}_1)\in \mathrm{W}^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})\times \mathrm{W}^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})$. Furthermore, $(\underline{u},\underline{v})\le(\underline{u}_1,\underline{v}_1)\le(\overline{u}_1,\overline{v}_1)\le(\overline{u},\overline{v})$.

Indeed, let $w_1=\overline{u}-\overline{u}_1$ and $w_2=\overline{v}-\overline{v}_1$. By (A3), (LL) and the definition of super-solution one gets

$$
\begin{gathered}
\frac{\partial w_1}{\partial t} - L_1 w_1 + \delta w_1 \ge 0 \quad \text{in } \Omega\times\mathbb{R},\\
\mathcal{B}_1 w_1 + \delta w_1 \ge 0 \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|w_1(x,t)|\} < \infty.
\end{gathered}
$$

and

$$
\begin{gathered}
\frac{\partial w_2}{\partial t} - L_2 w_2 + \delta w_2 \ge 0 \quad \text{in } \Omega\times\mathbb{R},\\
\mathcal{B}_2 w_2 + \delta w_2 \ge 0 \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|w_2(x,t)|\} < \infty.
\end{gathered}
$$

By Corollary 3.2 it follows that $w_1\ge 0$ and $w_2\ge 0$; that is, $\overline{u}_1\le\overline{u}$ and $\overline{v}_1\le\overline{v}$. Using the definition of subsolution we show in a similar way that $\underline{u}\le\underline{u}_1$ and $\underline{v}\le\underline{v}_1$.

We now need to prove that $\underline{u}_1 \le \overline{u}_1$ and $\underline{v}_1 \le \overline{v}_1$. Let $w_1 = \overline{u}_1 - \underline{u}_1$ and $w_2 = \overline{v}_1 - \underline{v}_1$. By (A3), (LL), (3.7), and the quasimonotone property, we have

$$\begin{aligned}\frac{\partial w_1}{\partial t} - L_1 w_1 + \delta w_1 &= [f_1(x,t,\overline{u},\overline{v}) + \delta\overline{u}] - [f_1(x,t,\underline{u},\underline{v}) + \delta\underline{u}] \\ &= [\delta(\overline{u} - \underline{u}) + f_1(x,t,\overline{u},\overline{v}) - f_1(x,t,\overline{u},\underline{v})] \\ &\quad + [f_1(x,t,\overline{u},\underline{v}) - f_1(x,t,\underline{u},\underline{v})] \\ &\ge 0 \text{ in } \Omega\times\mathbb{R},\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_1 w_1 + \delta w_1 &= [g_1(x,t,\overline{u},\overline{v}) + \delta\overline{u}] - [g_1(x,t,\underline{u},\underline{v}) + \delta\underline{u}] \\ &= [\delta(\overline{u} - \underline{u}) + g_1(x,t,\overline{u},\overline{v}) - g_1(x,t,\overline{u},\underline{v})] \\ &+ [g_1(x,t,\overline{u},\underline{v}) - g_1(x,t,\underline{u},\underline{v})] \\ &\ge 0 \quad \text{on } \partial\Omega\times\mathbb{R}.\end{aligned}$$

Since $\sup_{\Omega\times\mathbb{R}}\{|w_1(x,t)|\} < \infty$, it follows from Corollary 3.2 that $w_1 \ge 0$; that is, $\underline{u}_1 \le \overline{u}_1$. In a similar way, we prove that $\underline{v}_1 \le \overline{v}_1$.

For $n \ge 2$, a similar argument shows that depending on the choice of (u_0, v_0), problem (3.7) has solution either $(\underline{u}_n, \underline{v}_n)$ or $(\underline{u}_n, \underline{v}_n) \mathrm{W}^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R}) \times \mathrm{W}^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$ which is such that $(\underline{u}_{n-1}, \underline{v}_{n-1}) \le (u_n, v_n) \le (\overline{u}_{n-1}, \overline{v}_{n-1})$.

Indeed, assume by induction that for some $n \ge 2$,

$$\begin{gathered}\underline{u}_{n-1} \le \underline{u}_n \le \overline{u}_n \le \overline{u}_{n-1}, \\ \underline{v}_{n-1} \le \underline{v}_n \le \overline{v}_n \le \overline{v}_{n-1}\end{gathered}$$

Then, by (A3), (LL) and the quasimonotonicity property, the functions $w_1 = \overline{u}_{n+1} - \underline{u}_{n+1}$ and $w_2 = \overline{v}_{n+1} - \underline{v}_{n+1}$ satisfy

$$\begin{gathered}\frac{\partial w_1}{\partial t} - L_1 w_1 + \delta w_1 = [f_1(x,t,\overline{u}_n,\overline{v}_n) + \delta\overline{u}_n] - [f_1(x,t,\underline{u}_n,\underline{v}_n) + \delta\underline{u}_n] \ge 0 \\ \text{in } \Omega\times\mathbb{R}, \\ \mathcal{B}_1 w_1 + \delta w_1 = [g_1(x,t,\overline{u}_n,\overline{v}_n) + \delta\overline{u}_n] - [g_1(x,t,\underline{u}_n,\underline{v}_n) + \delta\underline{u}_n] \ge 0 \quad \text{on } \partial\Omega\times\mathbb{R}, \\ \sup_{\Omega\times\mathbb{R}}\{|w_1(x,t)|\} < \infty; \\ \frac{\partial w_2}{\partial t} - L_2 w_2 + \delta w_2 = [f_2(x,t,\overline{u}_n,\overline{v}_n) + \delta\overline{v}_n] - [f_2(x,t,\underline{u}_n,\underline{v}_n) + \delta\underline{v}_n] \ge 0 \\ \text{in } \Omega\times\mathbb{R}, \\ \mathcal{B}_2 w_2 + \delta w_2 = [g_2(x,t,\overline{u}_n,\overline{v}_n) + \delta\overline{v}_n] - [g_2(x,t,\underline{u}_n,\underline{v}_n) + \delta\underline{v}_n] \ge 0 \quad \text{on } \partial\Omega\times\mathbb{R}, \\ \sup_{\Omega\times\mathbb{R}}\{|w_2(x,t)|\} < \infty.\end{gathered}$$

Using Corollary 3.2 we get that $w_i \ge 0$ $(i = 1, 2)$; that is, $\underline{u}_{n+1} \le \overline{u}_{n+1}$ and $\underline{v}_{n+1} \le \overline{v}_{n+1}$. Using a similar argument as above we have that $\underline{u}_n \le \underline{u}_{n+1}$, $\overline{u}_{n+1} \le \overline{u}_n$, $\underline{v}_n \le \underline{v}_{n+1}$, and $\overline{v}_{n+1} \le \overline{v}_n$. Thus,

$$\begin{gathered}\underline{u} = \underline{u}_0 \le \underline{u}_1 \le \cdots \le \underline{u}_n \le \cdots \le \overline{u}_n \le \cdots \le \overline{u}_1 \le \overline{u}_0 = \overline{u} \\ \underline{v} = \underline{v}_0 \le \underline{v}_1 \le \cdots \le \underline{v}_n \le \cdots \le \overline{v}_n \le \cdots \le \overline{v}_1 \le \overline{v}_0 = \overline{v}\end{gathered}$$

Since the sequences $\{u_n\}$ and $\{v_n\}$ (where are u_n represents either $\underline{u}_n$ or $\overline{u}_n$ and v_n represents either $\underline{v}_n$ or $\overline{v}_n$) are monotone then the pointwise limits

$$u^*(x,t) = \lim_{n\to\infty} u_n(x,t) \quad \text{and} \quad v^*(x,t) = \lim_{n\to\infty} v_n(x,t)$$

both exist and $\underline{u} \le u^* \le \overline{u}$, and $\underline{v} \le v^* \le \overline{v}$. We now proceed to show that (u^*, v^*) is a solution of (3.6). For that purpose, consider $Q_1 = \Omega \times (-1,1)$ and $Q_2 = \Omega \times (-2,2)$. For each $n \in \mathbb{N}$, define $z_n(x,t) = \zeta(t)u_n(x,t)$, $w_n(x,t) = \zeta(t)v_n(x,t)$, for all $(x,t) \in \overline{\Omega} \times [-2,2]$, where $\zeta \in C^\infty(\mathbb{R}), 0 \le \zeta \le 1$ and $\zeta(s) = 0$ if $s \le -2$, $\zeta(s) = 1$ if $s \ge -(2-\delta)$ with $0 < \delta < 1$. Observe that $z_n = u_n$ and $w_n = v_n$, in $\overline{\Omega} \times [-1,1]$, and satisfy the linear uncoupled system

$$\begin{gathered}
\frac{\partial z_n}{\partial t} - L_1 z_n + \delta z_n = \frac{d\zeta}{dt} u_n + \zeta F_{1n} \quad \text{in } \Omega \times (-2,2], \\
\frac{\partial w_n}{\partial t} - L_2 w_n + \delta w_n = \frac{d\zeta}{dt} v_n + \zeta F_{2n} \quad \text{in } \Omega \times (-2,2], \\
\mathcal{B}_1 z_n + \delta z_n = \zeta G_{1n} \quad \text{on } \partial\Omega \times (-2,2], \\
\mathcal{B}_2 w_n + \delta w_n = \zeta G_{2n} \quad \text{on } \partial\Omega \times (-2,2], \\
z_n(x,-2) = 0 \quad \text{in } \overline{\Omega}, \\
w_n(x,-2) = 0 \quad \text{in } \overline{\Omega}, \\
\sup_{\Omega\times\mathbb{R}} \{|z_n(x,t)|, |w_n(x,t)|\} < \infty,
\end{gathered} \tag{3.8}$$

where

$$\begin{gathered}
F_{1n} = f_1(x,t,u_{n-1},v_{n-1}) + \delta u_{n-1}, \quad F_{2n} = f_2(x,t,u_{n-1},v_{n-1}) + \delta v_{n-1}, \\
G_{1n} = g_1(x,t,u_{n-1},v_{n-1}) + \delta u_{n-1}, \quad G_{2n} = g_2(x,t,u_{n-1},v_{n-1}) + \delta v_{n-1}.
\end{gathered}$$

By the solvability results on linear IBVPs with smooth coefficients [13, pp. 341-343], it follows that the linear problem (3.8) have a unique solution $(z_n, w_n) \in \mathrm{W}_p^{2,1}(Q_2) \times \mathrm{W}_p^{2,1}(Q_2)$ (with $p = \frac{N+2}{1-\mu}$). Moreover

$$|z_n|_{W_p^{2,1}(Q_2)} \le K_0\Big(|\frac{d\zeta}{dt}u_n + \zeta F_{1n}|_{L^p(Q_2)} + |\zeta G_{1n}|_{W_p^{2-\epsilon-\frac{1}{p},(2-\epsilon-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}\Big) \tag{3.9}$$

$$|w_n|_{W_p^{2,1}(Q_2)} \le K_0\Big(|\frac{d\zeta}{dt}v_n + \zeta F_{2n}|_{L^p(Q_2)} + |\zeta G_{2n}|_{W_p^{2-\epsilon-\frac{1}{p},(2-\epsilon-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}\Big), \tag{3.10}$$

for all $n \in \mathbb{N}$, where K_0 is a constant which depends on Q_2. Set $V_n = (z_n, w_n)$ with

$$|V_n|_{W_p^{2,1}(Q_2)} = |z_n|_{W_p^{2,1}(Q_2)} + |w_n|_{W_p^{2,1}(Q_2)}.$$

Observe that for $\epsilon = 0$, we get immediately that $|V_n|_{W_p^{2,1}(Q_2)} \le C$, for all n, since φ_0 does not depend on n. To show that $|V_n|_{W_p^{2,1}(Q_2)} \le C$ for all n for $\epsilon = 1$, we proceed as follows. Using assumptions (A2) we compute $|\zeta G_{i\,n}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}$ $(i = 1,2)$ to get that

$$|\zeta G_{i\,n}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))} \le \hat{C}\Big(1 + |V_{n-1}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}\Big), \tag{3.11}$$

where $\hat{C}$ is independent of n since $|\zeta G_{i\,n}|_{L^p(\partial\Omega\times(-2,2))} \le$ const for all $n \in \mathbb{N}$. Combining (3.9), (3.10), (3.11) we obtain

$$|V_n|_{W_p^{2,1}(Q_2)} \le \tilde{C}\Big(1 + |V_{n-1}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}\Big),$$

where $\tilde{C}$ is independent of n, but depends on $|\frac{d\zeta}{dt}v_n + \zeta F_{in}|_{L^p(Q_2)}$, $|\zeta G_{in}|_{L^p(\partial\Omega\times(-2,2))}$ and the set $\overline{\Omega}\times[-2,2]$. Using the continuity of the trace operator, we deduce that

$$|V_n|_{W_p^{2,1}(Q_2)} \le K\Big(1 + |V_{n-1}|_{W_p^{1,1/2}(\Omega\times(-2,2))}\Big), \tag{3.12}$$

where K does not depend on n. By the interpolation inequality (3.3), we get that

$$|V_n|_{W_p^{2,1}(Q_2)} \le K\Big(1 + C\varepsilon|V_{n-1}|_{W_p^{2,1}(Q_2)} + \frac{C}{4\varepsilon}|V_{n-1}|_{L^p(Q_2)}\Big) \tag{3.13}$$

From (3.12) we deduce that

$$|V_1|_{W_p^{2,1}(Q_2)} \le K\Big(1 + |\zeta\overline{V}|_{W_p^{1,1/2}(\Omega\times(-2,2))}\Big), \tag{3.14}$$

where $\overline{V}$ is either $(\underline{u},\underline{v})$ or $(\overline{u},\overline{v})$. Combining (3.13) with (3.14) we get

$$\begin{aligned}
|V_2|_{W_p^{2,1}(Q_2)} &\le K\Big(1 + C\varepsilon|V_1|_{W_p^{2,1}(X)} + \frac{C}{4\varepsilon}|V_1|_{L^p(Q_2)}\Big)\\
&\le K\Big(1 + KC\varepsilon + KC\varepsilon|\zeta\overline{V}|_{W_p^{1,1/2}(Q_2)} + \frac{C}{4\varepsilon}|V_1|_{L^p(Q_2)}\Big)
\end{aligned}$$

Proceeding by induction we have that for every $n \in \mathbb{N}$ with $n \ge 2$,

$$\begin{aligned}
|V_n|_{W_p^{2,1}(Q_2)} \le K\Big(&\sum_{i=0}^{n-1}(KC\varepsilon)^i + (KC\varepsilon)^{n-1}|\zeta\overline{V}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}\\
&+ \frac{MC}{4\varepsilon}\sum_{i=0}^{n-2}(KC\varepsilon)^i\Big),
\end{aligned}$$

where K is independent of n, and the constant $M \ge |V_n|_{L^p(Q_2)}$ for all $n \in \mathbb{N}$. Therefore, we obtain the following estimate which involves a geometric series

$$|V_n|_{W_p^{2,1}(Q_2)} \le \Big(K + K|\zeta\overline{V}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))} + \frac{MCK}{4\varepsilon}\Big)\sum_{i=0}^{\infty}(KC\varepsilon)^i.$$

Thus, $|V_n|_{W_p^{2,1}(Q_2)} \le C$ for all $n \in \mathbb{N}$, provided ε is chosen sufficiently small such that $KC\varepsilon < 1$. It follows that $|z_n|_{W_p^{2,1}(Q_2)} \le C$, $|w_n|_{W_p^{2,1}(Q_2)} \le C$.

Now, we need to show that in Q_1, the sequence $\{V_n\} = \{(u_n, v_n)\}$ has a subsequence which converges to a solution of problem (3.6). Indeed, define $T : \big(W_p^{2,1}(Q_1), |.|_{W_p^{2,1}(Q_1)}\big) \to (L^p(Q_1), |.|_{L^p(Q_1)})$ by

$$T(v) = \frac{\partial v}{\partial t} - Lv + \delta v.$$

Hence, T is (weakly) closed. Since $W_p^{2,1}(Q_1)$ is a reflexive space and $|V_n|_{W_p^{2,1}(Q_1)} \le \tilde{C}$ for all n, there exist subsequences $\{u_n\}$ and $\{v_n\}$ such that

$$u_n \rightharpoonup \tilde{u}_1 \quad\text{and}\quad v_n \rightharpoonup \tilde{v}_1 \quad\text{in } W_p^{2,1}(Q_1).$$

By the compact embedding of $W_p^{2,1}(Q_1)$ into $C^{1+\mu,(1+\mu)/2}(\overline{Q}_1)$, it follows that there exist subsequences $\{u_{1n}\}$ and $\{v_{1n'}\}$ such that $u_{1n} \to \tilde{u}_1$, $v_{1n'} \to \tilde{v}_1$ in $C^{1+\mu,(1+\mu)/2}(\overline{Q}_1)$. Since v_n is a monotone sequence, it follows that the sequence

$v_n \to \tilde{v}_1$ uniformly in $\overline{Q}_1$. Therefore, any subsequence of $\{v_n\}$ converges uniformly in $\overline{Q}_1$ to $\tilde{v}_1$ in particular $\{v_{1n}\}$. Moreover, since T is (weakly) closed and $T(u_{1n}) = F_{1n} \to f_1(.,.,\tilde{u}_1,\tilde{v}_1) + k\,\tilde{u}_1$ uniformly in $\overline{Q}_1$, it follows that $T(\tilde{u}_1) = f_1(.,.,\tilde{u}_1,\tilde{v}_1) + k\,\tilde{u}_1$. In addition, $\mathcal{B}_1 u_{1n} + \epsilon k\, u_{1n} \to \mathcal{B}_1\tilde{u}_1 + \epsilon k\,\tilde{u}_1$ in $C^{\mu,\mu/2}(\partial\Omega \times [-1,1])$, and $\mathcal{B}_1 u_{1n} + \epsilon k\, u_{1n} = G_{1n} \to g_1(.,.,\tilde{u}_1,\tilde{v}_1) + k\,\tilde{u}_1$ uniformly on $\partial\Omega \times [-1,1]$, we get $\mathcal{B}_1\tilde{u}_1 + \epsilon k\,\tilde{u}_1 = g_1(.,.,\tilde{u}_1,\tilde{v}_1) + k\,\tilde{u}_1$. Therefore, $\tilde{u}_1$ satisfies the following problem

$$
\begin{gathered}
\frac{\partial \tilde{u}_1}{\partial t} - L\tilde{u}_1 + k\tilde{u}_1 = f_1(x,t,\tilde{u}_1,\tilde{v}_1) + k\,\tilde{u}_1 \quad \text{in } \Omega \times (-1,1),\\
\mathcal{B}_1\tilde{u}_1 + \epsilon k\,\tilde{u}_1 = g_1(x,t,\tilde{u}_1,\tilde{v}_1) + k\,\tilde{u}_1 \quad \text{on } \partial\Omega \times [-1,1],\\
\sup_{\Omega\times[-1,1]} |\tilde{u}_1(x,t)| < \infty.
\end{gathered}
$$

Using similar arguments as above for v_n, we obtain

$$
\begin{gathered}
\frac{\partial \tilde{v}_1}{\partial t} - L\tilde{v}_1 + k\tilde{v}_1 = f_2(x,t,\tilde{u}_1,\tilde{v}_1) + k\,\tilde{v}_1 \quad \text{in } \Omega \times (-1,1),\\
\mathcal{B}_1\tilde{v}_1 + \epsilon k\,\tilde{v}_1 = g_2(x,t,\tilde{u}_1,\tilde{v}_1) + k\,\tilde{v}_1 \quad \text{on } \partial\Omega \times [-1,1],\\
\sup_{\Omega\times[-1,1]} |\tilde{v}_1(x,t)| < \infty.
\end{gathered}
$$

For $n \geq 2$, let $Q_n = \Omega \times (-n,n)$. Consider the subsequence $\{(u_{(n-1)k}, v_{(n-1)k})\}$ and use similar arguments to the above to extract a subsequence $\{(u_{n\,k}, v_{n\,k})\}$ of $\{(u_{(n-1)k}, v_{(n-1)k})\}$ such that it converges to $(\tilde{u}_n,\tilde{v}_n)$ in $C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-n,n]) \times C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-n,n])$ which satisfies

$$
\begin{gathered}
\frac{\partial \tilde{u}_n}{\partial t} - L\tilde{u}_n + k\tilde{u}_n = f_1(x,t,\tilde{u}_n,\tilde{v}_n) + k\tilde{u}_n \quad \text{in } \Omega \times (-n,n),\\
\frac{\partial \tilde{v}_n}{\partial t} - L\tilde{v}_n + k\tilde{v}_n = f_2(x,t,\tilde{u}_n,\tilde{v}_n) + k\,\tilde{v}_n \quad \text{in } \Omega \times (-n,n),\\
\mathcal{B}_1\tilde{u}_n + \epsilon k\,\tilde{u}_n = g_2(x,t,\tilde{u}_n,\tilde{v}_n) + k\tilde{u}_n \quad \text{on } \partial\Omega \times [-n,n],\\
\mathcal{B}_1\tilde{v}_n + \epsilon k\,\tilde{v}_n = g_2(x,t,\tilde{u}_n,\tilde{v}_n) + k\tilde{v}_n \quad \text{on } \partial\Omega \times [-n,n],\\
\sup_{\Omega\times[-n,n]} |\tilde{u}_n(x,t), \tilde{v}_n(x,t)| < \infty.
\end{gathered}
$$

Note that by construction, $(u_n,v_n)|_{\Omega\times[-(n-1),n-1]} = (u_{n-1},v_{n-1})$ for all $n \geq 2$; that is, (u_n,v_n) is an extension of (u_{n-1},v_{n-1}). Now, set $V_n = \{(u_n,v_n)\}$. By the diagonalization argument, choose the sequence $\{V_{jj}\}$ located on the 'diagonal'. Observe that $V_{jj} \in \{V_{nk}\} = \{(u_{nk},v_{nk})\}$ for every $n \leq j$, and hence $\{V_{jj}\}$ is a subsequence of $\{V_n\}$. We shall prove that the sequence $\{V_{jj}\}$ converges to a solution V^* of problem (3.6). Indeed, let $\overline{\Omega}\times[-n,n]$ and $\varepsilon > 0$. Since $\{V_{nk}\}$ converges to $\tilde{V}_n = (\tilde{u}_n,\tilde{v}_n)$ in $C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-n,n])$, there exists $N \in \mathbb{N}$ such that for all $k \geq N, |V_{nk} - \tilde{V}_k|_{C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-n,n])} < \varepsilon$. Using the fact that $V_{jj} \in \{V_{nk}\}$ for all $j \geq n$, we get that for all $j \geq \max\{n,N\}$, $|V_{jj} - \tilde{V}_n|_{C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-n,n])} < \varepsilon$.

Thus, $\{V_{jj}\}$ is subsequence of $\{V_n\}$ which converges (on every compact set) to a function $\tilde{V}$ in $C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-n,n])$, where $\tilde{V}|_{\overline{\Omega}\times[-n,n]} = \tilde{V}_n$, so that $\tilde{V} \in C^{1+\mu,(1+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R}) \cap W^{2,1}_{p,\rm loc}(\Omega\times\mathbb{R})$ and $\sup_{\Omega\times\mathbb{R}} |\tilde{V}| \leq M$. Moreover, $\tilde{V} = (\tilde{u},\tilde{v})$ satisfies the problem (3.6). By uniqueness of the limit we get $\tilde{V} = V^* = (u^*,v^*)$. Thus, (u^*,v^*) is a solution of problem (1.1) and $(\underline{u},\underline{v}) \leq (u^*,v^*) \leq (\overline{u},\overline{v})$. The proof is complete. $\square$

To prove Theorem 2.2, we will use (an improved version of) a nonlinear approximation argument inspired by the one considered in [15] (see also [1]). However, the main difficulty lies in the obtainment of required a priori estimates since there is a lack of compactness herein. We will therefore need the preliminary lemmas that as proved below. For sake of discussion, we will present the rest of the proof for the quasimonotone nonincreasing case.

Lemma 3.8. *Let f_1, f_2 satisfy* (A1)*; that is, $f_1,\ f_2 \in L^\infty_{\rm car}(\Omega\times\mathbb{R}\times\mathbb{R})$. Then, for every $r\in\mathbb{R}$ with $r>0$, there are continuous functions $m_1, m_2 : [-r,r]\times[-r,r]\to\mathbb{R}$ such that $m_i(\cdot,v)$ is nondecreasing on $[-r,r]$, $m_i(u,\cdot)$ is nonincreasing on $[-r,r]$, $m_i(u,v) = -m_i(v,u)$ on $[-r,r]\times[-r,r]$, and*

$$\sup_{\Omega\times\mathbb{R}} |f_1(x,t,u,w) - f_1(x,t,v,w)| \le m_1(u,v) \tag{3.15}$$

for all $u,v\in[-r,r]$ with $u\ge v$, and

$$\sup_{\Omega\times\mathbb{R}} |f_2(x,t,w,u) - f_2(x,t,w,v)| \le m_2(u,v) \tag{3.16}$$

for all $u,v\in[-r,r]$ with $u\ge v$.

The proof of Lemma 3.6 is similar to [15, Lemma 3.4]. Setting (for instance)

$$r = \max\left(|\underline{u}|_{L^\infty(\Omega\times\mathbb{R})}, |\overline{u}|_{L^\infty(\Omega\times\mathbb{R})}, \underline{u}|_{L^\infty(\Omega\times\mathbb{R})}, \overline{u}|_{L^\infty(\Omega\times\mathbb{R})}\right) + 2,$$

it follows from the Stone-Weierstrass Approximation Theorem that for every $n\in\mathbb{N}$ there is a Lipschitz continuous function $m_{i,n} : [-r,r]\times[-r,r]\to\mathbb{R}$ such that

$$|m_i(u,v) - m_{i,n}(u,v)| < \frac{1}{n} \tag{3.17}$$

for all $(u,v)\in[-r,r]\times[-r,r]$.

Now, consider the modified problems

$$\begin{gathered}
\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = f_1(x,t,\underline{u},\underline{v}) + m_1(\underline{u},u) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
\frac{\partial v}{\partial t}(x,t) - Lv(x,t) = f_2(x,t,\underline{u},\underline{v}) + m_2(\underline{v},v) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
\mathcal{B}_\epsilon u = g_1(x,t,\underline{u},\underline{v}) + \rho_1(\underline{u}-u) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\mathcal{B}_\epsilon v = g_2(x,t,\underline{u},\underline{v}) + \rho_2(\underline{v}-v) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|u(x,t)|, |v(x,t)|\} < \infty,
\end{gathered} \tag{3.18}$$

and

$$\begin{gathered}
\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = f_1(x,t,\overline{u},\overline{v}) + m_1(\overline{u},u) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
\frac{\partial v}{\partial t}(x,t) - Lv(x,t) = f_2(x,t,\overline{u},\overline{v}) + m_2(\overline{v},v) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
\mathcal{B}_1 u = g_1(x,t,\overline{u},\overline{v}) + \varrho_1(\overline{u}-u) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\mathcal{B}_2 v = g_2(x,t,\overline{u},\overline{v}) + \varrho_2(\overline{v}-v) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|u(x,t)|, |v(x,t)|\} < \infty.
\end{gathered} \tag{3.19}$$

Define the functions $\hat{f}_i, \check{f}_i \in L^\infty_{\rm car}(\Omega\times\mathbb{R}\times[\underline{u},\overline{u}])$ $(i=1,2)$ by

$$\hat{f}_1(x,t,u,\underline{v}) := f_1(x,t,\underline{u},\underline{v}) + m_1(\underline{u},u);$$

$$\begin{aligned}
\hat{f}_2(x,t,\underline{u},v) &:= f_2(x,t,\underline{u},\underline{v}) + m_2(\underline{v},v)\\
\check{f}_1(x,t,u,\overline{v}) &:= f_1(x,t,\overline{u},\overline{v}) + m_1(\overline{u},u);\\
\check{f}_2(x,t,\overline{u},v) &:= f_2(x,t,\overline{u},\overline{v}) + m_2(\overline{v},v).
\end{aligned}$$

Define the functions $\hat{g}_i$ $\check{g}_i$, which satisfy condition (A2), by

$$\begin{aligned}
\hat{g}_1(x,t,u,\underline{v}) &:= g_1(x,t,\underline{u},\underline{v}) + \rho_1(\underline{u}-u);\\
\hat{g}_2(x,t,\underline{u},v) &:= g_2(x,t,\underline{u},\underline{v}) + \rho_2(\underline{v}-v)\\
\check{g}_1(x,t,u,\overline{v}) &:= g_1(x,t,\overline{u},\overline{v}) + \varrho_1(\overline{u}-u);\\
\check{g}_2(x,t,\overline{u},v) &:= g_2(x,t,\overline{u},\overline{v}) + \varrho_2(\overline{v}-v).
\end{aligned}$$

Observe that $\hat{f}_1, \hat{g}_1$ ($\hat{f}_2, \hat{g}_2$) are nonincreasing in $u \in (-\infty, \underline{u}]$ (in $v \in (-\infty, \underline{v}]$), and $\check{f}_1, \check{g}_1$ ($\check{f}_2, \check{g}_2$) are nonincreasing in $u \in [\overline{u}, \infty)$ (in $v \in [\underline{v}, \infty)$). Moreover, by using Lemma 3.6, (3.15) and (3.16), they satisfy the following inequalities:

$$\begin{aligned}
\hat{f}_1(x,t,\cdot,\underline{v}) \le f_1(x,t,\cdot,\underline{v}) \le \check{f}_1(x,t,\cdot,\overline{v});\\
\hat{g}_1(x,t,\cdot,\underline{v}) \le g_1(x,t,\cdot,\underline{v}) \le \check{g}_1(x,t,\cdot,\overline{v})
\end{aligned}$$

on $[\underline{u}, \overline{u}]$.

$$\begin{aligned}
\hat{f}_2(x,t,\underline{u},\cdot) \le \hat{f}_2(x,t,\underline{u},\cdot) \le \check{f}_2(x,t,\overline{u},\cdot);\\
\hat{g}_2(x,t,\underline{u},\cdot) \le \hat{g}_2(x,t,\underline{u},\cdot) \le \check{g}_2(x,t,\overline{u},\cdot)
\end{aligned}$$

on $[\underline{v}, \overline{v}]$.

We will show that problem (3.18) and problem (3.19) have unique solutions in $[(\underline{u},\underline{v}), (\overline{u},\overline{v})]$. In order to accomplish this, we first need the following lemma.

Lemma 3.9. *Assume that* (A1)–(A3) *are satisfied and that* $(\underline{u},\underline{v})$ *and* $(\overline{u},\overline{v})$ *are subsolution and supersolution of problem* (1.1) *with* $\underline{u} \le \overline{u}$ *and* $\underline{v} \le \overline{v}$ *. Let* $\delta > 0$ *and* $\overline{u}_\delta := \overline{u} + \delta z$*,* $\underline{u}_\delta := \underline{u} - \delta z$*,* $\overline{v}_\delta := \overline{v} + \delta z$*,* $\underline{v}_\delta := \underline{v} - \delta z$*, where* z *is the (unique) solution of the linear boundary value problem*

$$\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = 1 \quad \text{a.e in} \quad \Omega\times\mathbb{R},\ \mathcal{B}u = 1+\epsilon \quad \text{on } \partial\Omega\times\mathbb{R}, \tag{3.20}$$

and $\sup_{\Omega\times\mathbb{R}} |u(x,t)| < \infty$*. Then the boundary value problems*

$$\begin{gathered}
\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = f(x,t,\underline{u},\underline{v}) + m_1(\underline{u}_\delta, u) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
\frac{\partial v}{\partial t}(x,t) - Lv(x,t) = f(x,t,\underline{u},\underline{v}) + m_2(\underline{v}_\delta, v) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
\mathcal{B}_1 u = g_1(x,t,\underline{u},\underline{v}) + \varrho_1(\underline{u}_\delta - u) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\mathcal{B}_2 v = g_2(x,t,\underline{u},\underline{v}) + \varrho_2(\underline{v}_\delta - v) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|u(x,t)|, |v(x,t)|\} < \infty,
\end{gathered} \tag{3.21}$$

and

$$\begin{gathered}\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = f(x,t,\overline{u},\overline{v}) + m_1(\overline{u}_\delta, u) \quad a.e.\ in\ \Omega\times\mathbb{R},\\ \frac{\partial v}{\partial t}(x,t) - Lv(x,t) = f(x,t,\overline{u},\overline{v}) + m_2(\overline{v}_\delta, v) \quad a.e.\ in\ \Omega\times\mathbb{R},\\ \mathcal{B}_1 u = g_1(x,t,\overline{u},\overline{v}) + \varrho_1(\overline{u}_\delta - u) \quad on\ \partial\Omega\times\mathbb{R},\\ \mathcal{B}_2 v = g_2(x,t,\overline{u},\overline{v}) + \varrho_2(\overline{v}_\delta - v) \quad on\ \partial\Omega\times\mathbb{R},\\ \sup_{\Omega\times\mathbb{R}}\{|u(x,t)|,|v(x,t)|\} < \infty,\end{gathered}\tag{3.22}$$

have unique solutions $(\hat{u}_\delta, \hat{v}_\delta)$ *and* $(\check{u}_\delta, \check{v}_\delta)$ *respectively such that*

$$\underline{u}_\delta \le \hat{u}_\delta \le \check{u}_\delta \le \overline{u}_\delta \quad and \quad \underline{v}_\delta \le \hat{v}_\delta \le \check{v}_\delta \le \overline{v}_\delta.$$

Proof. Let us define the functions

$$\begin{gathered}\hat{f}_{1\delta}(x,t,u,\underline{v}) := f_1(x,t,\underline{u},\underline{v}) + m_1(\underline{u}_\delta, u);\\ \hat{f}_{2\delta}(x,t,\underline{u},v) := f_2(x,t,\underline{u},\underline{v}) + m_2(\underline{v}_\delta, v)\\ \check{f}_{1\delta}(x,t,u,\overline{v}) := f_1(x,t,\overline{u},\overline{v}) + m_1(\overline{u}_\delta, u);\\ \check{f}_{2\delta}(x,t,\overline{u},v) := f_2(x,t,\overline{u},\overline{v}) + m_2(\overline{v}_\delta, v).\\ \hat{g}_{1\delta}(x,t,u,\underline{v}) := g_1(x,t,\underline{u},\underline{v}) + \rho_1(\underline{u}_\delta - u);\\ \hat{g}_{2\delta}(x,t,\underline{u},v) := g_2(x,t,\underline{u},\underline{v}) + \rho_2(\underline{v}_\delta - v);\\ \check{g}_{1\delta}(x,t,u,\overline{v}) := g_1(x,t,\overline{u},\overline{v}) + \varrho_1(\overline{u}_\delta - u);\\ \check{g}_{2\delta}(x,t,\overline{u},v) := g_2(x,t,\overline{u},\overline{v}) + \varrho_2(\overline{v}_\delta - v).\end{gathered}$$

From the monotonicity properties of m_i, one has that $\hat{f}_{1\delta}, \hat{g}_{1\delta}$ ($\hat{f}_{2\delta}, \hat{g}_{2\delta}$) are nonincreasing functions of u for $u \ge \underline{u}_\delta$ (of v for $v \ge \underline{v}_\delta$), and $\check{f}_{1\delta}, \check{g}_{1\delta}$ ($\check{f}_{2\delta}, \check{g}_{2\delta}$) are nonincreasing functions of u for $u \le \overline{u}_\delta$ (of v for $v \le \overline{v}_\delta$).

Using the definitions and the monotonicity properties of m_i, the quasimonotonicity property of f_i, and the fact that $\hat{f}_1(x,t,\cdot,\underline{v}) \le \check{f}_1(x,t,\cdot,\overline{v})$ on $[\underline{u},\overline{u}]$ and $\hat{f}_2(x,t,\underline{u},\cdot) \le \check{f}_2(x,t,\overline{u},\cdot)$ on $[\underline{v},\overline{v}]$, it follows that

$$\hat{f}_{1\delta}(x,t,u,\underline{v}) \le \check{f}_{1\delta}(x,t,u,\overline{v}) \quad \text{for all } u \in [\underline{u}_\delta, \overline{u}_\delta]\tag{3.23}$$

and

$$\hat{f}_{2\delta}(x,t,\underline{u},v) \le \check{f}_{2\delta}(x,t,\overline{u},v) \quad \text{for all } v \in [\underline{v}_\delta, \overline{v}_\delta].\tag{3.24}$$

Now, by using (3.20) and the fact that $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are sub- and super-solutions, it is seen that

$$\begin{gathered}\frac{\partial \underline{u}_\delta}{\partial t}(x,t) - L\underline{u}_\delta(x,t) - \hat{f}_{1\delta}(x,t,\underline{u}_\delta,\underline{v}) \le -\delta \quad \text{a.e. in } \Omega\times\mathbb{R},\\ \frac{\partial \underline{v}_\delta}{\partial t}(x,t) - L\underline{v}_\delta(x,t) - \hat{f}_{2\delta}(x,t,\underline{u},\underline{v}_\delta) \le -\delta \quad \text{a.e. in } \Omega\times\mathbb{R},\\ \mathcal{B}_1\underline{u}_\delta - \hat{g}_{1\delta}(x,t,\underline{u}_\delta,\underline{v}) \le -2\delta \quad \text{on } \partial\Omega\times\mathbb{R},\\ \mathcal{B}_2\underline{v}_\delta - \hat{g}_{2\delta}(x,t,\underline{u},\underline{v}_\delta) \le -2\delta \quad \text{on } \partial\Omega\times\mathbb{R},\\ \sup_{\Omega\times\mathbb{R}}\{|\underline{u}_\delta(x,t)|,|\underline{v}_\delta(x,t)|\} < \infty,\end{gathered}\tag{3.25}$$

and

$$
\begin{aligned}
&\frac{\partial \overline{u}_\delta}{\partial t}(x,t) - L\overline{u}_\delta(x,t) - \check{f}_{1\delta}(x,t,\overline{u}_\delta,\overline{v}) \geq \delta \quad \text{a.e. in } \Omega\times\mathbb{R},\\
&\frac{\partial \overline{v}_\delta}{\partial t}(x,t) - L\overline{v}_\delta(x,t) - \check{f}_{2\delta}(x,t,\overline{u},\overline{v}_\delta) \geq \delta \quad \text{a.e. in } \Omega\times\mathbb{R},\\
&\mathcal{B}_1\overline{u}_\delta - \check{g}_{1\delta}(x,t,\overline{u}_\delta,\overline{v}) \geq 2\delta \quad \text{on } \partial\Omega\times\mathbb{R},\\
&\mathcal{B}_2\overline{v}_\delta - \check{g}_{2\delta}(x,t,\overline{u},\overline{v}_\delta) \geq 2\delta \quad \text{on } \partial\Omega\times\mathbb{R},\\
&\sup_{\Omega\times\mathbb{R}}\{|\overline{u}_\delta(x,t)|, |\overline{v}_\delta(x,t)|\} < \infty.
\end{aligned} \tag{3.26}
$$

Therefore $(\underline{u}_\delta, \underline{v}_\delta)$ is a strict subsolution of problem (3.21) and $(\overline{u}_\delta, \overline{v}_\delta)$ is a strict supersolution of problem (3.22). The functions $\hat{f}_1, \hat{f}_2, \check{f}_1, \check{f}_2, \hat{g}_1, \hat{g}_2, \check{g}_1$, and $\check{g}_2$ defined on $\overline{\Omega}\times\mathbb{R}\times[\underline{u}_\delta, \overline{u}_\delta]\times[\underline{v}_\delta, \overline{v}_\delta]$ may be approximated, for $n\in\mathbb{N}$, by the following functions

$$
\begin{aligned}
\hat{f}_{1n}(x,t,u,\underline{v}) &:= f_1(x,t,\underline{u},\underline{v}) + m_{1n}(\underline{u}_\delta, u);\\
\hat{f}_{2n}(x,t,\underline{u},v) &:= f_2(x,t,\underline{u},\underline{v}) + m_{2n}(\underline{v}_\delta, v)\\
\check{f}_{1n}(x,t,u,\overline{v}) &:= f_1(x,t,\overline{u},\overline{v}) + m_{1n}(\overline{u}_\delta, u);\\
\check{f}_{2n}(x,t,\overline{u},v) &:= f_2(x,t,\overline{u},\overline{v}) + m_{2n}(\overline{v}_\delta, v).\\
\hat{g}_{1n}(x,t,u,\underline{v}) &:= g_1(x,t,\underline{u},\underline{v}) + \rho_{1n}(\underline{u}_\delta - u);\\
\hat{g}_{2n}(x,t,\underline{u},v) &:= g_2(x,t,\underline{u},\underline{v}) + \rho_{2n}(\underline{v}_\delta - v)\\
\check{g}_{1n}(x,t,u,\overline{v}) &:= g_1(x,t,\overline{u},\overline{v}) + \varrho_{1n}(\overline{u}_\delta - u);\\
\check{g}_{2n}(x,t,\overline{u},v) &:= g_2(x,t,\overline{u},\overline{v}) + \varrho_{2n}(\overline{v}_\delta - v),
\end{aligned}
$$

where m_{in} are the Lipschitz approximation of m_i satisfying (3.17). Moreover, it is easy to check that $(\underline{u}_\delta, \underline{v}_\delta)$ and $(\overline{u}_\delta, \overline{v}_\delta)$ are still supersolution and subsolution of the following approximating equations

$$
\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = f_1(x,t,\underline{u},\underline{v}) + m_{1n}(\underline{u}_\delta, u) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
&\frac{\partial v}{\partial t}(x,t) - Lv(x,t) = f_2(x,t,\underline{u},\underline{v}) + m_{2n}(\underline{v}_\delta, v) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
&\mathcal{B}_1 u = g_1(x,t,\underline{u},\underline{v}) + \varrho_1(\underline{u}_\delta - u) \quad \text{on } \partial\Omega\times\mathbb{R},\\
&\mathcal{B}_2 v = g_2(x,t,\underline{u},\underline{v}) + \varrho_2(\underline{v}_\delta - v) \quad \text{on } \partial\Omega\times\mathbb{R},\\
&\sup_{\Omega\times\mathbb{R}}\{|u(x,t)|, |v(x,t)|\} < \infty,
\end{aligned} \tag{3.27}
$$

and

$$
\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = f_1(x,t,\overline{u},\overline{v}) + m_{1n}(\overline{u}_\delta, u) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
&\frac{\partial v}{\partial t}(x,t) - Lv(x,t) = f_2(x,t,\overline{u},\overline{v}) + m_{2n}(\overline{v}_\delta, v) \quad \text{a.e. in } \Omega\times\mathbb{R},\\
&\mathcal{B}_1 u = g_1(x,t,\overline{u},\overline{v}) + \varrho_1(\overline{u}_\delta - u) \quad \text{on } \partial\Omega\times\mathbb{R},\\
&\mathcal{B}_2 v = g_2(x,t,\overline{u},\overline{v}) + \varrho_2(\overline{v}_\delta - v) \quad \text{on } \partial\Omega\times\mathbb{R},\\
&\sup_{\Omega\times\mathbb{R}}\{|u(x,t)|, |v(x,t)|\} < \infty.
\end{aligned} \tag{3.28}
$$

Since $(\underline{u}_\delta, \underline{v}_\delta) \leq (\overline{u}_\delta, \overline{v}_\delta)$ and the functions $\hat{f}_{in}$ and $\check{f}_{in}$ satisfy the (LL)-condition in $[\underline{u}_\delta, \overline{u}_\delta]\times[\underline{v}_\delta, \overline{v}_\delta]$, it follows from Proposition 3.7 that there is a solution $(\hat{u}_{\delta,n}, \hat{v}_{\delta,n})$

of problem (3.27) and a solution $(\check{u}_{\delta n}, \check{v}_{\delta n})$ of problem (3.28) such that $(\underline{u}_\delta, \underline{v}_\delta) \le (\hat{u}_{\delta n}, \hat{v}_{\delta n}) \le (\overline{u}_\delta, \overline{v}_\delta)$ and $(\underline{u}_\delta, \underline{v}_\delta) \le (\check{u}_{\delta n}, \check{v}_{\delta n}) \le (\overline{u}_\delta, \overline{v}_\delta)$.

Now we proceed to show that (a relabeled subsequence of) $(\hat{u}_{\delta n}, \hat{v}_{\delta n})$ converges (uniformly on compact sets) to a solution $(\hat{u}_\delta, \hat{v}_\delta)$ of problem (3.21) with $(\underline{u}_\delta, \underline{v}_\delta) \le (\hat{u}_\delta, \hat{v}_\delta) \le (\overline{u}_\delta, \overline{v}_\delta)$.

For that purpose, consider $Q_1 = \Omega \times (-1, 1)$ and $Q_2 = \Omega \times (-2, 2)$. For each $n \in \mathbb{N}$, define $\hat{z}_n(x,t) = \zeta(t)\hat{u}_{\delta,n}(x,t)$, $\hat{w}_n(x,t) = \zeta(t)\hat{v}_{\delta,n}(x,t)$, for all $(x,t) \in \overline{\Omega} \times [-2, 2]$, where $\zeta \in C^\infty(\mathbb{R}), 0 \le \zeta \le 1$ and $\zeta(s) = 0$ if $s \le -2$, $\zeta(s) = 1$ if $s \ge -(2-\delta)$ with $0 < \delta < 1$. Observe that $z_n = u_n$ and $w_n = v_n$, in $\overline{\Omega} \times [-1, 1]$, and satisfy the linear uncoupled system

$$
\begin{gathered}
\frac{\partial \hat{z}_n}{\partial t} - L_1 \hat{z}_n = \frac{d\zeta}{dt} \hat{u}_{\delta n} + \zeta \hat{f}_{1n} \quad \text{in } \Omega \times (-2, 2], \\
\frac{\partial \hat{w}_n}{\partial t} - L_2 \hat{w}_n = \frac{d\zeta}{dt} \hat{v}_{\delta n} + \zeta \hat{f}_{2n} \quad \text{in } \Omega \times (-2, 2], \\
\mathcal{B}_1 \hat{z}_n = \zeta \hat{g}_{1n} \quad \text{on } \partial\Omega \times (-2, 2], \\
\mathcal{B}_2 \hat{w}_n = \zeta \hat{g}_{2n} \quad \text{on } \partial\Omega \times (-2, 2], \\
\hat{z}_n(x, -2) = 0 \quad \text{in } \overline{\Omega}, \\
\hat{w}_n(x, -2) = 0 \quad \text{in } \overline{\Omega}, \\
\sup_{\Omega \times \mathbb{R}} \{|\hat{z}_n(x,t)|, |\hat{w}_n(x,t)|\} < \infty.
\end{gathered} \tag{3.29}
$$

Using arguments similar to the proof of Proposition 3.7 we show that (a subsequence relabeled as) $(\hat{u}_{\delta n}, \hat{v}_{\delta n})$ converges (on compact sets) to a solution $(\hat{u}_\delta, \hat{v}_\delta)$ of problem (3.21) with $(\underline{u}_\delta, \underline{v}_\delta) \le (\hat{u}_\delta, \hat{v}_\delta) \le (\overline{u}_\delta, \overline{v}_\delta)$. Likewise, (a subsequence of) $(\check{u}_{\delta n}, \check{v}_{\delta n})$ converges (on compact sets) to a solution $(\check{u}_\delta, \check{v}_\delta)$ of problem (3.21) with $(\underline{u}_\delta, \underline{v}_\delta) \le (\check{u}_\delta, \check{v}_\delta) \le (\overline{u}_\delta, \overline{v}_\delta)$. Observe that, by (3.23) and (3.24), $\check{f}_{1\delta}(x, t, \check{u}_\delta, \overline{v}) \ge \hat{f}_{1\delta}(x, t, \check{u}_\delta, \underline{v})$, $\check{f}_{2\delta}(x, t, \overline{u}, \check{v}_\delta) \ge \hat{f}_{2\delta}(x, t, \underline{v}, \check{v}_\delta)$, $\check{g}_{1\delta}(x, t, \check{u}_\delta, \overline{v}) \ge \hat{g}_{1\delta}(x, t, \check{u}_\delta, \underline{v})$, and $\check{g}_{2\delta}(x, t, \overline{u}, \check{v}_\delta) \ge \hat{g}_{2\delta}(x, t, \underline{v}, \check{v}_\delta)$. Therefore,

$$
\begin{gathered}
\frac{\partial(\check{u}_\delta - \hat{u}_\delta)}{\partial t} - L(\check{u}_\delta - \hat{u}_\delta) \ge \hat{f}_{1\delta}(x, t, \check{u}_\delta, \overline{v}) - \hat{f}_{1\delta}(x, t, \hat{u}_\delta, \underline{v}) \quad \text{a.e. in } \Omega \times \mathbb{R}, \\
\mathcal{B}_\epsilon(\check{u}_\delta - \hat{u}_\delta) \ge \hat{g}_{1\delta}(x, t, \check{u}_\delta, \overline{v}) - \hat{g}_{1\delta}(x, t, \hat{u}_\delta, \underline{v}) \quad \text{on } \partial\Omega \times \mathbb{R}, \\
\frac{\partial(\check{v}_\delta - \hat{v}_\delta)}{\partial t} - L(\check{v}_\delta - \hat{v}_\delta) \ge \hat{f}_{2\delta}(x, t, \overline{u}, \check{v}_\delta) - \hat{f}_{2,\delta}(x, t, \underline{v}, \hat{v}_\delta) \quad \text{a.e. in } \Omega \times \mathbb{R}, \\
\mathcal{B}_\epsilon(\check{v}_\delta - \hat{v}_\delta) \ge \hat{g}_{2\delta}(x, t, \overline{u}, \check{v}_\delta) - \hat{g}_{2\delta}(x, t, \underline{v}, \hat{v}_\delta) \quad \text{on } \partial\Omega \times \mathbb{R}.
\end{gathered}
$$

The monotonicity of the functions $\hat{f}_{i\delta}$, $\hat{g}_{i\delta}$, $\check{f}_{i\delta}$ and $\check{g}_{i\delta}$, and an argument similar to the one used in the proof of [15, Proposition 2.7] imply that $\hat{u}_\delta \le \check{u}_\delta$ and $\hat{v}_\delta \le \check{v}_\delta$ in $\overline{\Omega} \times \mathbb{R}$. The proof is complete. □

Lemma 3.10. *Assume that* (A1)–(A3) *are satisfied and that* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *are subsolution and supersolution of problem* (1.1) *with* $\underline{u} \le \overline{u}$ *and* $\underline{v} \le \overline{v}$. *Then there exist unique solutions* $(\hat{u}, \hat{v}), (\check{u}, \check{v}) \in W^{2,1}_{p,\mathrm{loc}}(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R}) \times W^{2,1}_{p,\mathrm{loc}}(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$ *to the respective problems* (3.18) *and* (3.19) *such that*

$$
\underline{u} \le \hat{u} \le \check{u} \le \overline{u} \quad \text{and} \quad \underline{v} \le \hat{v} \le \check{v} \le \overline{v}.
$$

Proof. Since the systems (3.18) and (3.19) are uncoupled, we have that the uniqueness follows from the (nonincreasing) monotonicity of the nonlinearities involved and Proposition 3.4. For $n \in \mathbb{N}$, let $\underline{u}_n = \underline{u} - \frac{1}{n} z$, $\underline{v}_n = \underline{v} - \frac{1}{n} z$, $\overline{u}_n = \overline{u} + \frac{1}{n} z$, and $\overline{v}_n = \overline{v} + \frac{1}{n}z$, where z is defined in Lemma 3.9. Consider the boundary value problems (3.21) and (3.22) where the right hand sides are replaced by

$$\begin{aligned}
\hat{f}_{1n}(x,t,u,\underline{v}) &:= f_1(x,t,\underline{u},\underline{v}) + m_1(\underline{u}_n,u);\\
\hat{f}_{2n}(x,t,\underline{u},v) &:= f_2(x,t,\underline{u},\underline{v}) + m_2(\underline{v}_n,v)\\
\check{f}_{1n}(x,t,u,\overline{v}) &:= f_1(x,t,\overline{u},\overline{v}) + m_1(\overline{u}_n,u);\\
\check{f}_{2n}(x,t,\overline{u},v) &:= f_2(x,t,\overline{u},\overline{v}) + m_2(\overline{v}_n,v)\\
\hat{g}_{1n}(x,t,u,\underline{v}) &:= g_1(x,t,\underline{u},\underline{v}) + \rho_1(\underline{u}_n - u);\\
\hat{g}_{2n}(x,t,\underline{u},v) &:= g_2(x,t,\underline{u},\underline{v}) + \rho_2(\underline{v}_n - v)\\
\check{g}_{1n}(x,t,u,\overline{v}) &:= g_1(x,t,\overline{u},\overline{v}) + \varrho_1(\overline{u}_n - u);\\
\check{g}_{2n}(x,t,\overline{u},v) &:= g_2(x,t,\overline{u},\overline{v}) + \varrho_2(\overline{v}_n - v),
\end{aligned}$$

respectively. By applying Lemma 3.9 with $\delta = \frac{1}{n}$, we get that, for each $n \in \mathbb{N}$ there exist unique solutions $(\hat{u}_n, \hat{v}_n)$, $(\check{u}_n, \check{v}_n) \in W^{2,1}_{p,\mathrm{loc}}(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R}) \times W^{2,1}_{p,\mathrm{loc}}(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$ of these corresponding problems such that

$$\underline{u}_n \le \hat{u}_n \le \check{u}_n \le \overline{u}_n \quad \text{and} \quad \underline{v}_n \le \hat{v}_n \le \check{v}_n \le \overline{v}_n.$$

Next, we show that the sequences $\{\hat{u}_n\}$, $\{\hat{v}_n\}$, $\{\check{u}_n\}$, and $\{\check{v}_n\}$ are monotone and converge to unique solutions of (3.18) and (3.19), respectively. From the definitions of $\underline{u}_n$, $\underline{v}_n$, $\overline{u}_n$, and $\overline{v}_n$ we have that the functions $\hat{f}_{1n}(x,t,\cdot,\underline{v})$ and $\hat{g}_{1n}(x,t,\cdot,\underline{v})$ are nonincreasing for $u \in [\underline{u}_n, \infty)$, and $\hat{f}_{2n}(x,t,\underline{u},\cdot)$ and $\hat{g}_{2n}(x,t,\underline{u},\cdot)$ are nonincreasing for $v \in [\underline{v}_n, \infty)$, and that they are nondecreasing with respect to n. Similarly, the functions $\check{f}_{1n}(x,t,\cdot,\overline{v})$ are nonincreasing for $u \in (-\infty, \overline{v}_n]$, and $\check{g}_{1n}(x,t,\cdot,\overline{v})$ are nonincreasing for $\check{f}_{2n}(x,t,\overline{u},\cdot)$ and $\check{g}_{2n}(x,t,\overline{u},\cdot)$ are nonincreasing for $v \in (-\infty, \overline{v}_n]$, and they are nonincreasing with respect to n. Therefore,

$$\begin{aligned}
\frac{\partial(\hat{u}_n - \hat{u}_{n-1})}{\partial t} - L(\hat{u}_n - \hat{u}_{n-1}) &\ge \hat{f}_{1(n-1)}(x,t,\hat{u}_n,\underline{v}) - \hat{f}_{1(n-1)}(x,t,\hat{u}_{n-1},\underline{v}),\\
\mathcal{B}_\epsilon(\hat{u}_n - \hat{u}_{n-1}) &\ge \hat{g}_{1(n-1)}(x,t,\hat{u}_n,\underline{v}) - \hat{g}_{1(n-1)}(x,t,\hat{u}_{n-1},\underline{v}),\\
\frac{\partial(\hat{v}_n - \hat{v}_{n-1})}{\partial t} - L(\hat{v}_n - \hat{v}_{n-1}) &\ge \hat{f}_{2(n-1)}(x,t,\underline{u},\hat{v}_n) - \hat{f}_{2(n-1)}(x,t,\underline{u},\hat{v}_{n-1}),\\
\mathcal{B}_\epsilon(\hat{v}_n - \hat{v}_{n-1}) &\ge \hat{g}_{2(n-1)}(x,t,\underline{u},\hat{v}_n) - \hat{g}_{2(n-1)}(x,t,\underline{u},\hat{v}_{n-1}).
\end{aligned}$$

By the monotonicity of $\hat{f}_{1n}(x,t,\cdot,\underline{v}), \hat{g}_{1n}(x,t,\cdot,\underline{v})$, $\hat{f}_{2n}(x,t,\underline{u},\cdot)$, $\hat{g}_{2n}(x,t,\underline{u},\cdot)$ and Corollary 3.2, we show as in the proof of Lemma 3.9 that $\hat{u}_{n-1} \le \hat{u}_n$ and $\hat{v}_{n-1} \le \hat{v}_n$. Similarly, we get that $\check{u}_n \le \check{u}_{n-1}$ and $\check{v}_n \le \check{v}_{n-1}$. Therefore,

$$\begin{gathered}
\underline{u}_1 \le \hat{u}_1 \le \hat{u}_2 \le \dots \le \hat{u}_{n-1} \le \hat{u}_n \le \ldots \le \check{u}_n \le \check{u}_{n-1} \le \ldots \le \check{u}_2 \le \check{u}_1 \le \overline{u}_1.\\
\underline{v}_1 \le \hat{v}_1 \le \hat{v}_2 \le \dots \le \hat{v}_{n-1} \le \hat{v}_n \le \ldots \le \check{v}_n \le \check{v}_{n-1} \le \ldots \le \check{v}_2 \le \check{v}_1 \le \overline{v}_1.
\end{gathered}$$

It follows that $\{(\hat{u}_n, \hat{v}_n)\}$ and $\{(\check{u}_n, \check{v}_n)\}$ converge (pointwise) to $(\hat{u}, \hat{v})$ and $(\check{u}, \check{v})$, respectively, with $(\underline{u}, \underline{v}) \le (\hat{u}, \hat{v}) \le (\check{u}, \check{v}) \le (\overline{u}, \overline{v})$. Now, we will show that $\underline{v}$ and $\overline{v}$ are respective solutions of (3.18) and (3.19). Consider $Q_1 = \Omega \times (-1,1)$ and $Q_2 = \Omega \times (-2,2)$. For each $n \in \mathbb{N}$, define $\hat{z}_n(x,t) = \zeta(t)\hat{u}_n(x,t)$, $\hat{w}_n(x,t) = \zeta(t)\hat{v}_n(x,t)$, for all $(x,t) \in \overline{\Omega} \times [-2,2]$, where $\zeta \in C^\infty(\mathbb{R}), 0 \le \zeta \le 1$ and $\zeta(s) = 0$ if $s \le -2, \zeta(s) = 1$ if $s \ge -(2-\delta)$ with $0 < \delta < 1$. Observe that $\hat{z}_n = \hat{u}_n$ and

$\hat{w}_n = \hat{v}_n$, in $\overline{\Omega} \times [-1,1]$, and satisfy the linear uncoupled system

$$\begin{gathered}
\frac{\partial \hat{z}_n}{\partial t} - L_1 \hat{z}_n = \frac{d\zeta}{dt}\hat{u}_n + \zeta \hat{f}_{1n}(x,t,\hat{u}_n,\underline{v}) \quad \text{in } \Omega \times (-2,2], \\
\frac{\partial \hat{w}_n}{\partial t} - L_2 \hat{w}_n = \frac{d\zeta}{dt}\hat{v}_n + \zeta \hat{f}_{2n}(x,t,\overline{u},\hat{v}_n) \quad \text{in } \Omega \times (-2,2], \\
\mathcal{B}_1 \hat{z}_n = \zeta \hat{g}_{1n}(x,t,\hat{u}_n,\underline{v}) \quad \text{on } \partial\Omega \times (-2,2], \\
\mathcal{B}_2 \hat{w}_n = \zeta \hat{g}_{2n}(x,t,\overline{u},\hat{v}_n) \quad \text{on } \partial\Omega \times (-2,2], \\
\hat{z}_n(x,-2) = 0 \quad \text{in } \overline{\Omega}, \\
\hat{w}_n(x,-2) = 0 \quad \text{in } \overline{\Omega}, \\
\sup_{\Omega \times \mathbb{R}} \{|\hat{z}_n(x,t)|, |\hat{w}_n(x,t)|\} < \infty.
\end{gathered} \tag{3.30}$$

Arguments similar to those used in the proof of Proposition 3.7 show that $(\hat{u},\hat{v})$ and $(\check{u},\check{v}) \in W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R}) \times W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$ are solutions of (3.18) and (3.19), respectively, with $(\underline{u},\underline{v}) \le (\hat{u},\hat{v}) \le (\check{u},\check{v}) \le (\overline{u},\overline{v})$. The proof is complete. □

Proof of Theorem 2.2. We construct two sequences $\{(\hat{u}_n,\hat{v}_n)\}$ and $\{(\check{u}_n,\check{v}_n)\}$ successively from the nonlinear iteration processes

$$\begin{gathered}
\frac{\partial \hat{u}_n}{\partial t}(x,t) - L\hat{u}_n(x,t) = f_1(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) + m_1(\hat{u}_{n-1},\hat{u}_n) \quad \text{a.e. in } \Omega\times\mathbb{R}, \\
\frac{\partial \hat{v}_n}{\partial t}(x,t) - L\hat{v}_n(x,t) = f_2(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) + m_2(\hat{v}_{n-1},\hat{v}_n) \quad \text{a.e. in } \Omega\times\mathbb{R}, \\
\mathcal{B}_1 \hat{u}_n = g_1(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) + \varrho_1(\hat{u}_{n-1} - \hat{u}_n) \quad \text{on } \partial\Omega\times\mathbb{R}, \\
\mathcal{B}_2 \hat{v}_n = g_2(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) + \varrho_2(\hat{v}_{n-1} - \hat{v}_n) \quad \text{on } \partial\Omega\times\mathbb{R}, \\
\sup_{\Omega\times\mathbb{R}} \{|\hat{u}_n(x,t)|, |\hat{v}_n(x,t)|\} < \infty,
\end{gathered} \tag{3.31}$$

and

$$\begin{gathered}
\frac{\partial \check{u}_n}{\partial t}(x,t) - L\check{u}_n(x,t) = f_1(x,t,\check{u}_{n-1},\check{v}_{n-1}) + m_1(\check{u}_{n-1},\check{u}_n) \quad \text{a.e. in } \Omega\times\mathbb{R}, \\
\frac{\partial \check{v}_n}{\partial t}(x,t) - L\check{v}_n(x,t) = f_2(x,t,\check{u}_{n-1},\check{v}_{n-1}) + m_2(\check{v}_{n-1},\check{v}_n) \quad \text{a.e. in } \Omega\times\mathbb{R}, \\
\mathcal{B}_1 \check{u}_n = g_1(x,t,\check{u}_{n-1},\check{v}_{n-1}) + \varrho_1(\check{u}_{n-1} - \check{u}_n) \quad \text{on } \partial\Omega\times\mathbb{R}, \\
\mathcal{B}_2 \check{v}_n = g_2(x,t,\check{u}_{n-1},\check{v}_{n-1}) + \varrho_2(\check{v}_{n-1} - \check{v}_n) \quad \text{on } \partial\Omega\times\mathbb{R}, \\
\sup_{\Omega\times\mathbb{R}} \{|\check{u}_n(x,t)|, |\check{v}_n(x,t)|\} < \infty.
\end{gathered} \tag{3.32}$$

We show that these sequences are well defined and that they converge monotonically to a solution of (1.1). Indeed, set

$$\begin{gathered}
\hat{f}_{1n}(x,t,u,\hat{v}_{n-1}) = f_1(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) + m_1(\hat{u}_{n-1},u), \\
\hat{f}_{2n}(x,t,\hat{u}_{n-1},v) = f_2(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) + m_2(\hat{v}_{n-1},v), \\
\hat{g}_{1n}(x,t,u,\hat{v}_{n-1}) = f_1(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) + m_1(\hat{u}_{n-1},u), \\
\hat{g}_{2n}(x,t,\hat{u}_{n-1},v) = f_2(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) + m_2(\hat{v}_{n-1},v),
\end{gathered}$$

$$
\begin{aligned}
\check{f}_{1n}(x,t,u,\check{v}_{n-1}) &= f_1(x,t,\check{u}_{n-1},\check{v}_{n-1}) + m_1(\check{u}_{n-1},u),\\
\check{f}_{2n}(x,t,\check{u}_{n-1},v) &= f_2(x,t,\check{u}_{n-1},\check{v}_{n-1}) + m_2(\check{v}_{n-1},v),\\
\check{g}_{1n}(x,t,u,\check{v}_{n-1}) &= f_1(x,t,\check{u}_{n-1},\check{v}_{n-1}) + m_1(\check{u}_{n-1},u),\\
\check{g}_{2n}(x,t,\check{u}_{n-1},v) &= f_2(x,t,\check{u}_{n-1},\check{v}_{n-1}) + m_2(\check{v}_{n-1},v),
\end{aligned}
$$

where $(\hat{u}_0,\hat{v}_0) = (\underline{u},\underline{v})$ and $(\check{u}_0,\check{v}_0) = (\overline{u},\overline{v})$. It follows immediately from Lemma 3.10 that the first iterations $(\hat{u}_1,\hat{v}_1)$ in (3.31) and $(\check{u}_1,\check{v}_1)$ in (3.32) exist and satisfy the inequalities $(\underline{u},\underline{v}) \le (\hat{u}_1,\hat{v}_1) \le (\check{u}_1,\check{v}) \le (\overline{u},\overline{v})$, when one starts with $(\hat{u}_0,\hat{v}_0) = (\underline{u},\underline{v})$ and $(\check{u}_0,\check{v}_0) = (\overline{u},\overline{v})$. For $n \ge 2$, we use an induction argument to show that $(\hat{u}_{n-1},\hat{v}_{n-1}) \le (\hat{u}_n,\hat{v}_n) \le (\check{u}_{n-1},\check{v}_{n-1}) \le (\check{u}_n,\check{v}_n)$. However, in order to apply Lemma 3.10 inductively, we need to show that, at each iteration, the functions $(\hat{u}_{n-1},\hat{v}_{n-1})$ and $(\check{u}_{n-1},\check{v}_{n-1})$ are ordered subsolution and supersolution of problem (1.1). By using (3.15), (A2), quasimonotonicity decreasing and the equations (3.31) and (3.32), we get

$$
\begin{gathered}
\begin{aligned}
&\frac{\partial \hat{u}_{n-1}}{\partial t} - L\hat{u}_{n-1} - f_1(x,t,\hat{u}_{n-1},\hat{v}_{n-1})\\
&= f_1(x,t,\hat{u}_{n-2},\hat{v}_{n-2}) + m_1(\hat{u}_{n-2},\hat{u}_{n-1}) - f_1(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) \le 0,\\
&\frac{\partial \hat{v}_{n-1}}{\partial t} - L\hat{v}_{n-1} - f_2(x,t,\hat{u}_{n-1},\hat{v}_{n-1})\\
&= f_2(x,t,\hat{u}_{n-2},\hat{v}_{n-2}) + m_2(\hat{v}_{n-2},\hat{v}_{n-1}) - f_2(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) \le 0,\\
&\mathcal{B}\hat{u}_{n-1} - g_1(x,t,\hat{u}_{n-1},\hat{v}_{n-1})\\
&= g_1(x,t,\hat{u}_{n-2},\hat{v}_{n-2}) + \varrho(\hat{u}_{n-2} - \hat{u}_{n-1}) - g_1(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) \le 0,\\
&\mathcal{B}\hat{v}_{n-1} - g_2(x,t,\hat{u}_{n-1},\hat{v}_{n-1})\\
&= g_2(x,t,\hat{u}_{n-2},\hat{v}_{n-2}) + \varrho(\hat{v}_{n-2} - \hat{v}_{n-1}) - g_2(x,t,\hat{u}_{n-1},\hat{v}_{n-1}) \le 0,
\end{aligned}\\
\sup_{\Omega\times\mathbb{R}}\{|\hat{u}_{n-1}(x,t)|,|\hat{v}_{n-1}(x,t)|\} < \infty,
\end{gathered}
$$

and

$$
\begin{gathered}
\begin{aligned}
&\frac{\partial \check{u}_{n-1}}{\partial t} - L\check{u}_{n-1} - f_1(x,t,\check{u}_{n-1},\check{v}_{n-1})\\
&= f_1(x,t,\check{u}_{n-2},\check{v}_{n-2}) + m_1(\check{u}_{n-2},\check{u}_{n-1}) - f_1(x,t,\check{u}_{n-1},\check{v}_{n-1}) \ge 0,\\
&\frac{\partial \check{v}_{n-1}}{\partial t} - L\check{v}_{n-1} - f_2(x,t,\check{u}_{n-1},\check{v}_{n-1})\\
&= f_2(x,t,\check{u}_{n-2},\check{v}_{n-2}) + m_2(\hat{v}_{n-2},\hat{v}_{n-1}) - f_2(x,t,\check{u}_{n-1},\check{v}_{n-1}) \ge 0,\\
&\mathcal{B}\check{u}_{n-1} - g_1(x,t,\check{u}_{n-1},\check{v}_{n-1})\\
&= g_1(x,t,\check{u}_{n-2},\check{v}_{n-2}) + \varrho_1(\check{u}_{n-2} - \check{u}_{n-1}) - g_1(x,t,\check{u}_{n-1},\check{v}_{n-1}) \ge 0,\\
&\mathcal{B}\check{v}_{n-1} - g_2(x,t,\check{u}_{n-1},\check{v}_{n-1})\\
&= g_2(x,t,\check{u}_{n-2},\check{v}_{n-2}) + \varrho_2(\check{v}_{n-2} - \check{v}_{n-1}) - g_2(x,t,\check{u}_{n-1},\check{v}_{n-1}) \ge 0,
\end{aligned}\\
\sup_{\Omega\times\mathbb{R}}\{|\check{u}_{n-1}(x,t)|,|\check{v}_{n-1}(x,t)|\} < \infty,
\end{gathered}
$$

which shows that, for $n \ge 2$, the functions $(\hat{u}_{n-1},\hat{v}_{n-1})$ and $(\check{u}_{n-1},\check{v}_{n-1})$ are ordered subsolution and supersolution of (1.1). Since $\hat{f}_{1n},\hat{f}_{2n}$, $\check{f}_{1n},\check{f}_{2n}$, and $\hat{g}_{1n},\hat{g}_{2n}$, $\check{g}_{1n},\check{g}_{2n}$ satisfy the conditions of Lemma 3.10 with $(\hat{u}_{n-1},\hat{v}_{n-1})$ as subsolution and $(\check{u}_{n-1},\check{v}_{n-1})$ as supersolution, the existence of solutions $(\hat{u}_n,\hat{v}_n)$ to Eq.(3.31) and $(\check{u}_n,\check{v}_n)$ to Eq.(3.32) such that $(\hat{u}_{n-1},\hat{v}_{n-1}) \le (\hat{u}_n,\hat{v}_n) \le (\check{u}_n,\check{v}_n) \le (\check{u}_{n-1},\check{v}_{n-1})$

is ensured by Lemma 3.10. We therefore have that

$$\begin{aligned}\underline{u}_1 \le \hat{u}_1 \le \hat{u}_2 \le \dots \le \hat{u}_{n-1} \le \hat{u}_n \le \ldots \le \check{u}_n \le \check{u}_{n-1} \le \ldots \le \check{u}_2 \le \check{u}_1 \le \overline{u}_1,\\ \underline{v}_1 \le \hat{v}_1 \le \hat{v}_2 \le \dots \le \hat{v}_{n-1} \le \hat{v}_n \le \ldots \le \check{v}_n \le \check{v}_{n-1} \le \ldots \le \check{v}_2 \le \check{v}_1 \le \overline{v}_1.\end{aligned}$$

It follows that $\{(\hat{u}_n,\hat{v}_n)\}$ and $\{(\check{u}_n,\check{v}_n)\}$ converge (pointwise) to $(\hat{u},\hat{v})$ and $(\check{u},\check{v})$, respectively, with $(\underline{u},\underline{v}) \le (\hat{u},\hat{v}) \le (\check{u},\check{v}) \le (\overline{u},\overline{v})$. Consider $Q_1 = \Omega\times(-1,1)$ and $Q_2 = \Omega\times(-2,2)$. For each $n\in\mathbb{N}$, define $\hat{z}_n(x,t) = \zeta(t)\hat{u}_n(x,t)$, $\hat{w}_n(x,t) = \zeta(t)\hat{v}_n(x,t)$, for all $(x,t)\in\overline{\Omega}\times[-2,2]$, where $\zeta\in C^\infty(\mathbb{R})$, $0\le\zeta\le 1$ and $\zeta(s)=0$ if $s\le -2$, $\zeta(s)=1$ if $s\ge -(2-\delta)$ with $0<\delta<1$. Observe that $\hat{z}_n = \hat{u}_n$ and $\hat{w}_n = \hat{v}_n$, in $\overline{\Omega}\times[-1,1]$ and (z_n,w_n) satisfies the following uncouple system

$$\begin{gathered}\frac{\partial \hat{z}_n}{\partial t} - L_1\hat{z}_n = \frac{d\zeta}{dt}\hat{u}_n + \zeta\hat{f}_{1n}(x,t,\hat{u}_n,\hat{v}_{n-1}) \quad \text{in } \Omega\times(-2,2],\\ \frac{\partial \hat{w}_n}{\partial t} - L_2\hat{w}_n = \frac{d\zeta}{dt}\hat{v}_n + \zeta\hat{f}_{2n}(x,t,\hat{u}_{n-1},\hat{v}_n) \quad \text{in } \Omega\times(-2,2],\\ \mathcal{B}_1\hat{z}_n = \zeta\hat{g}_{1n}(x,t,\hat{u}_n,\hat{v}_{n-1}) \quad \text{on } \partial\Omega\times(-2,2],\\ \mathcal{B}_2\hat{w}_n = \zeta\hat{g}_{2n}(x,t,\hat{u}_{n-1},\hat{v}_n) \quad \text{on } \partial\Omega\times(-2,2],\\ \hat{z}_n(x,-2) = 0 \quad \text{in } \overline{\Omega},\\ \hat{w}_n(x,-2) = 0 \quad \text{in } \overline{\Omega},\\ \sup_{\Omega\times\mathbb{R}}\{|\hat{z}_n(x,t)|,|\hat{w}_n(x,t)|\} < \infty,\end{gathered}$$

where $(z_n,\, w_n)\in \mathrm{W}^{2,1}_p(Q_2)\times\mathrm{W}^{2,1}_p(Q_2)$ (with $p=\frac{N+2}{1-\mu}$). Moreover

$$\begin{aligned}|z_n|_{W^{2,1}_p(Q_2)} &\le K_0\Big(|\frac{d\zeta}{dt}\hat{u}_n + \zeta\hat{f}_{1,n}|_{L^p(Q_2)} + |\zeta\hat{g}_{1,n}|_{W_p^{2-\epsilon-\frac1p,(2-\epsilon-\frac1p)/2}(\partial\Omega\times(-2,2))}\Big),\\ |w_n|_{W^{2,1}_p(Q_2)} &\le K_0\Big(|\frac{d\zeta}{dt}\hat{v}_n + \zeta\hat{f}_{2,n}|_{L^p(Q_2)} + |\zeta\hat{g}_{2n}|_{W_p^{2-\epsilon-\frac1p,(2-\epsilon-\frac1p)/2}(\partial\Omega\times(-2,2))}\Big),\end{aligned}$$

for all $n\in\mathbb{N}$, where K_0 is a constant which depends on Q_2. Set $V_n=(z_n,w_n)$ with $|V_n|_{W^{2,1}_p(Q_2)} = |z_n|_{W^{2,1}_p(Q_2)} + |w_n|_{W^{2,1}_p(Q_2)}$. Observe that for the Dirichlet boundary condition, we get immediately that $|V_n|_{W^{2,1}_p(Q_2)}\le C$, for all n. To show that $|V_n|_{W^{2,1}_p(Q_2)}\le C$ for all n for the Neumann boundary condition, we proceed as follows. Using assumptions (A2) we compute $|\zeta\hat{g}_{i\,n}|_{W_p^{1-\frac1p,(1-\frac1p)/2}(\partial\Omega\times(-2,2))}$ ($i=1,2$) to obtain

$$\begin{aligned}&|\zeta\hat{g}_{i\,n}|_{W_p^{1-\frac1p,(1-\frac1p)/2}(\partial\Omega\times(-2,2))}\\ &\le \hat{C}(1+|V_n|_{W_p^{1-\frac1p,(1-\frac1p)/2}(\partial\Omega\times(-2,2))} + |V_{n-1}|_{W_p^{1-\frac1p,(1-\frac1p)/2}(\partial\Omega\times(-2,2))}),\end{aligned}$$

where $\hat{C}$ is independent of n since $|\zeta\hat{g}_{i,n}|_{L^p(\partial\Omega\times(-2,2))}\le$ const for all $n\in\mathbb{N}$. Using the continuity of the trace operator and the fact that $\hat{u}_n,\hat{v}_n$ and $\hat{f}_{i,n}$ are (uniformly) bounded, we get that

$$|V_n|_{W^{2,1}_p(Q_2)} \le \tilde{K}\Big(1 + |V_n|_{W^{1,1/2}_p(Q_2)} + |V_{n-1}|_{W^{1,1/2(Q_2)}_p}\Big),$$

where K is independent of n. Using the interpolation inequality (3.3), we obtain

$$|V_n|_{W^{2,1}_p(Q_2)} \le \frac{K}{1-\varepsilon K}\Big(1 + \frac{1}{\varepsilon}|V_n|_{L^p(Q_2)} + |V_{n-1}|_{W^{1,1/2}_p(Q_2)}\Big),$$

where $\varepsilon\tilde{K} < 1$. Since V_n is (uniformly) bounded, we deduce that

$$|V_n|_{W_p^{2,1}(Q_2)} \le C\Big(1 + |V_{n-1}|_{W_p^{1,1/2}(Q_2)}\Big),$$

where C is a constant independent of n. Using the same reasoning as in the proof of Proposition 3.7, one shows that $\{V_n\}$ converges to a solution $(\hat{u},\hat{v})$ of (1.1) with $(\hat{u},\hat{v}) \in W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R}) \times W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$. An analogous argument shows also that $(\check{u},\check{v}) \in W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R}) \times W^{2,1}_{p,\mathrm{loc}}(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$ is a solution of (1.1), and that $(\underline{u},\underline{v}) \le (\hat{u},\hat{v}) \le (\check{u},\check{v}) \le (\overline{u},\overline{v})$. □

4. Examples

In this section, we illustrate our results with the following examples.

Example 4.1. (Cooperative Model) Consider the system

$$\begin{gathered}
\frac{\partial u}{\partial t} - \triangle u = u(a_1(x,t) - b_1(x,t)u + c_1(x,t)v) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial v}{\partial t} - \triangle v = v(a_2(x,t) + b_2(x,t)u - c_2(x,t)v) \quad \text{in } \Omega\times\mathbb{R},\\
u = 0 = v \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|u|,|v|\} < \infty,
\end{gathered} \tag{4.1}$$

where $a_i : \Omega\times\mathbb{R} \to \mathbb{R}$, $b_i : \Omega\times\mathbb{R} \to \mathbb{R}$, $c_i : \Omega\times\mathbb{R} \to \mathbb{R}$ are in $C^{\mu,\mu/2}_{\mathrm{loc}}(\overline{\Omega}\times\mathbb{R})$. The coefficients satisfy the following conditions: For all $(x,t) \in \Omega\times\mathbb{R}$, $\lambda_1 < \alpha_i \le a_i(x,t) \le A_i$, $0 < \beta_i \le b_i(x,t) \le B_i$,

$$0 < \gamma_i \le c_i(x,t) \le C_i, \quad \text{and} \quad \frac{B_2}{\beta_1} < \frac{\gamma_2}{C_1}.$$

Here, λ_1 is the first eigenvalue of the Laplacian, and $\alpha_i,\beta_i,\gamma_i,A_i,B_i,C_i \in \mathbb{R}$. Observe that the presence of the $u-$population species encourages the growth of the $v-$population species and vice versa. So, the reaction functions $f_1(x,t,u,v) = u(a_1(x,t) - b_1(x,t)u + c_1(x,t)v)$ and $f_2(x,t,u,v) = v(a_2(x,t) + b_2(x,t)u - c_2(x,t)v)$ are quasimonotone nondecreasing in $[0,\infty)\times[0,\infty)$. In order to apply Theorem 2.2, we need to have an ordered sub-solution $(\underline{u},\underline{v})$ and super-solution $(\overline{u},\overline{v})$ of (4.1) that satisfy the following inequalities

$$\begin{gathered}
\frac{\partial \underline{u}}{\partial t} - \triangle \underline{u} \le \underline{u}(a_1(x,t) - b_1(x,t)\underline{u} + c_1(x,t)\underline{v}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial \underline{v}}{\partial t} - \triangle \underline{v} \le \underline{v}(a_2(x,t) + b_2(x,t)\underline{u} - c_2(x,t)\underline{v}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial \overline{u}}{\partial t} - \triangle \overline{u} \ge \overline{u}(a_1(x,t) - b_1(x,t)\overline{u} + c_1(x,t)\overline{v}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial \overline{v}}{\partial t} - \triangle \overline{v} \ge \overline{v}(a_2(x,t) + b_2(x,t)\overline{u} - c_2(x,t)\overline{v} \quad \text{in } \Omega\times\mathbb{R},\\
\underline{u} \le 0 \le \overline{u} \quad \text{on } \partial\Omega\times\mathbb{R},\\
\underline{v} \le 0 \le \overline{v} \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|\underline{u}|,|\overline{u}|,|\underline{v}|,|\overline{v}|\} < \infty.
\end{gathered}$$

Choose $(\underline{u},\underline{v}) = (\varepsilon\varphi,\varepsilon\varphi)$ with $0 < \varepsilon < \min\{\frac{\alpha_1-\lambda_1}{B_1},\frac{\alpha_2-\lambda_1}{B_2}\}$ where $0 < \varphi$ is the eigenfunction associated with λ_1. Pick $(\overline{u},\overline{v}) = (M_1,M_2)$, where $M_1 = \tau\frac{A_1\gamma_2+A_2C_1}{\beta_1\gamma_2-B_2C_1}$

and $M_2 = \tau\frac{A_2\beta_1+A_1B_2}{\beta_1\gamma_2-B_2C_1}$. The positive constant τ is chosen so that $(\varepsilon\varphi,\varepsilon\varphi) \le (M_1, M_2)$. Then by Theorem 2.2, there exists a positive solution (u, v) such that $(\varepsilon\varphi,\varepsilon\varphi) \le (u, v) \le (M_1, M_2)$ in $\overline{\Omega}\times\mathbb{R}$. Thus, (u, v) does not tend to zero as t tends to $\pm\infty$, for each $x \in \Omega$.

Example 4.2. (Generalized Cooperative Model with nonlinear Boundary conditions)

$$\begin{gathered}
\frac{\partial u}{\partial t} - \triangle u = u^m(a_1(x,t) - b_1(x,t)u + c_1(x,t)v) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial v}{\partial t} - \triangle v = v^m(a_2(x,t) + b_2(x,t)u - c_2(x,t)v) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial u}{\partial \nu} = u^n\left(\delta_1 - u + \sigma_1 v\right) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\frac{\partial v}{\partial \nu} = v^n\left(\delta_2 - v + \sigma_2 u\right) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|u|,|v|\} < \infty,
\end{gathered} \tag{4.2}$$

where $n, m \in \mathbb{N}$, $0 < \delta_i \in \mathbb{R}$, $0 < \sigma_i < 1$,

$$a_i : \Omega\times\mathbb{R}\to\mathbb{R}, b_i : \Omega\times\mathbb{R}\to\mathbb{R}, c_i : \Omega\times\mathbb{R}\to\mathbb{R} \quad \text{are in } C^{\mu,\mu/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R}).$$

For all $(x, t) \in \Omega\times\mathbb{R}$, $0 < \alpha_i \le a_i(x, t) \le A_i$, $0 < \beta_i \le b_i(x, t) \le B_i$, $0 < \gamma_i \le c_i(x, t) \le C_i$, where $\alpha_i, \beta_i, \gamma_i, A_i, B_i, C_i \in \mathbb{R}$. Note that the nonlinearities satisfy the quasimonotone nondecreasing property. Choose $(\underline{u}, \underline{v}) = (D, D)$, where $D > 0$ is very small such that $D < \min\{\delta_1, \delta_2\}$. Pick $(\overline{u}, \overline{v}) = (M, M)$, where M is a constant such that $M > \max\left\{\frac{\delta_1}{1-\sigma_1}, \frac{\delta_2}{1-\sigma_2}\right\}$.
Then it follows from Theorem 2.2 that the system (4.2) has a positive solution (u, v) such that $(D, D) \le (u, v) \le (M, M)$ in $\overline{\Omega}\times\mathbb{R}$. Thus, (u, v) does not tend to zero as t tends to $\pm\infty$.

Example 4.3. (Competitive Model)

$$\begin{gathered}
\frac{\partial u}{\partial t} - \triangle u = u(a_1(x,t) - b_1(x,t)u - c_1(x,t)v) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial v}{\partial t} - \triangle v = v(a_2(x,t) - b_2(x,t)v - c_2(x,t)u) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|u|,|v|\} < \infty,
\end{gathered} \tag{4.3}$$

where $a_i : \Omega\times\mathbb{R}\to\mathbb{R}$, $b_i : \Omega\times\mathbb{R}\to\mathbb{R}$, $c_i : \Omega\times\mathbb{R}\to\mathbb{R}$ are in $C^{\mu,\mu/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})$. For all $(x, t) \in \Omega\times\mathbb{R}$, $0 < \alpha_i \le a_i(x, t) \le A_i$, $0 < \beta_i \le b_i(x, t) \le B_i$, $0 < \gamma_i \le c_i(x, t) \le C_i$, where $\alpha_i, \beta_i, \gamma_i, A_i, B_i, C_i \in \mathbb{R}$. Observe that under competition, the growth of each population is reduced at a rate proportional to the size of the population of its competitor. We are concerned with the existence of positive solutions for the problem (4.3), which can be translated as the co-existence of the two populations without asymptotic extinction.

Note that the reaction functions are quasimonotone nonincreasing in $[0,\infty)\times[0,\infty)$. To apply Theorem 2.2, we need to have an ordered sub-solution $(\underline{u}, \underline{v})$ and super-solution $(\overline{u},\overline{v})$ of (4.3). As in Definition 2.1, they need to satisfy the following inequalities

$$\begin{gathered}
\frac{\partial \underline{u}}{\partial t}-\triangle \underline{u} \leq \underline{u}(a_1(x,t)-b_1(x,t)\underline{u}-c_1(x,t)\overline{v}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial \overline{v}}{\partial t}-\triangle \overline{v} \geq \overline{v}(a_2(x,t)-b_2(x,t)\overline{v}-c_2(x,t)\underline{u}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial \overline{u}}{\partial t}-\triangle \overline{u} \geq \overline{u}(a_1(x,t)-b_1(x,t)\overline{u}-c_1(x,t)\underline{v}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial \underline{v}}{\partial t}-\triangle \underline{v} \leq \underline{v}(a_2(x,t)-b_2(x,t)\underline{v}-c_2(x,t)\overline{u}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial \underline{u}}{\partial \nu} \leq 0 \leq \frac{\partial \overline{u}}{\partial \nu} \quad \text{on } \partial\Omega\times\mathbb{R},\\
\frac{\partial \underline{v}}{\partial \nu} \leq 0 \leq \frac{\partial \overline{v}}{\partial \nu} \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|\underline{u}|,|\overline{u}|,|\underline{v}|,|\overline{v}|\}<\infty.
\end{gathered} \tag{4.4}$$

Choose $(\overline{u},\overline{v})=(M_1,M_2)$ such that $M_i \geq \frac{A_i}{\beta_i}$ $(i=1,2)$. Pick $(\underline{u},\underline{v})=(\varepsilon_1,\varepsilon_2)$ such that $\varepsilon_i \leq \frac{\alpha_i\beta_j-A_jC_i}{\beta_jB_i}$ with $C_i<\frac{\alpha_i\beta_j}{A_j}$ for $i,j=1,2$ and $(i\neq j)$. With this choice of sub- and super-solutions, one can see that $(\underline{u},\underline{v})\leq \overline{u},\overline{v})$ and inequalities in (4.4) are satisfied. Therefore, it follows from Theorem 2.2 that problem 4.3 has a positive solution (u,v) such that $(\varepsilon_1,\varepsilon_2)\leq(u,v)\leq(M_1,M_2)$ in $\overline{\Omega}\times\mathbb{R}$. Thus, (u,v) does not tend to zero as t tends to $\pm\infty$, for each $x\in\Omega$.

Example 4.4. (Competitive Model with Nonlinear Boundary Conditions)

$$\begin{gathered}
\frac{\partial u}{\partial t}-\triangle u = u(a_1(x,t)-b_1(x,t)u-c_1(x,t)v) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial v}{\partial t}-\triangle v = v(a_2(x,t)-b_2(x,t)v-c_2(x,t)u) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial u}{\partial \nu} = u^m(\delta_1-u-\sigma_1 v) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\frac{\partial v}{\partial \nu} = v^m(\delta_2-v-\sigma_2 u) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}\{|u|,|v|\}<\infty.
\end{gathered} \tag{4.5}$$

where $0<\delta_i,\sigma_i\in\mathbb{R}$, $a_i:\Omega\times\mathbb{R}\to\mathbb{R}$, $b_i:\Omega\times\mathbb{R}\to\mathbb{R}$, $c_i:\Omega\times\mathbb{R}\to\mathbb{R}$ are in $C^{\mu,\mu/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})$. For all $(x,t)\in\Omega\times\mathbb{R}$, $0<\alpha_i\leq a_i(x,t)\leq A_i$, $0<\beta_i\leq b_i(x,t)\leq B_i$, $0<\gamma_i\leq c_i(x,t)\leq C_i$, where $\alpha_i,\beta_i,\gamma_i,A_i,B_i,C_i\in\mathbb{R}$.

Note that the nonlinearities satisfy the quasimonotone nondecreasing property. Choose $(\overline{u},\overline{v})=(M_1,M_2)$ with $M_i\geq\delta_i+\frac{A_i}{\beta_i}(i=1,2)$, and pick $(\underline{u},\underline{v})=(\varepsilon_1,\varepsilon_2)$, where $0<\varepsilon_i<\min\{\delta_i-\sigma_iM_j,\ \alpha_i-C_iM_j\}$ with $C_i<\frac{\alpha_j}{\delta_j+\frac{A_j}{\beta_j}}$ $(i,j=1,2$ with $i\neq j)$. Then by Theorem 2.2, the system (4.5) has a positive solution (u,v) such that $(\varepsilon_1,\varepsilon_2)\leq(u,v)\leq(M_1,M_2)$ in $\overline{\Omega}\times\mathbb{R}$. Thus, (u,v) does not tend to zero as t tends to $\pm\infty$.

Example 4.5. (Nonlinearities with no one-sided Lipschitz conditions)

$$
\begin{gathered}
\frac{\partial u}{\partial t} - \triangle u = f_1(x,t,u,v) \quad \text{in } \Omega \times \mathbb{R}, \\
\frac{\partial v}{\partial t} - \triangle v = f_2(x,t,u,v) \quad \text{in } \Omega \times \mathbb{R}, \\
\frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega \times \mathbb{R}, \\
\sup_{\Omega \times \mathbb{R}} \{|u|, |v|\} < \infty.
\end{gathered}
\tag{4.6}
$$

where

$$
f_1(x,t,u,v) = \begin{cases} -b_1(x,t)u^\mu v & \text{if } 0 \le u, v < \infty, \text{ for some } 0 < \mu < 1 \\ 0 & \text{if } -\infty \le u, v \le 0 \end{cases}
$$

where

$$
f_2(x,t,u,v) = \begin{cases} -b_2(x,t)v^\mu u & \text{if } 0 \le u, v < \infty, \text{ for some } 0 < \mu < 1 \\ 0 & \text{if } -\infty \le u, v \le 0 \end{cases}
$$

and $b_i \in L^\infty(\Omega \times \mathbb{R})$ such that $0 < \beta \le b_i(x,t) \le B$ for a.e $(x,t) \in \Omega \times \mathbb{R}$, where $\beta, B \in \mathbb{R}$.

It is easily seen that $(\underline{u}, \underline{v}) = (0,0)$ and $(\overline{u}, \overline{v}) = (K,K)$ (where $0 < K \in \mathbb{R}$) are ordered subsolution and supersolution of problem (4.6). Therefore, by Theorem 2.2, there exists a solution (u,v) of problem (4.6) such that $(0,0) \le (u,v) \le (K,K)$.

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